

# ON LIPSCHITZ SOLUTIONS FOR SOME FORWARD-BACKWARD PARABOLIC EQUATIONS. II: THE CASE AGAINST FOURIER

SEONGHAK KIM AND BAISHENG YAN

**ABSTRACT.** As a sequel to the paper [9], we study the existence and properties of Lipschitz solutions to the initial-boundary value problem of some forward-backward parabolic equations with diffusion fluxes violating Fourier's inequality.

## 1. INTRODUCTION

In this paper, following the authors' recent result [9], we further investigate the initial-boundary value problem

$$(1.1) \quad \begin{cases} u_t = \operatorname{div}(A(Du)) & \text{in } \Omega_T, \\ A(Du) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T), \\ u = u_0 & \text{on } \Omega \times \{t = 0\}, \end{cases}$$

where  $\Omega \subset \mathbb{R}^n$  ( $n \geq 1$ ) is a bounded domain,  $T > 0$  is any fixed number,  $\Omega_T = \Omega \times (0, T)$ ,  $\mathbf{n}$  is the outer unit normal on  $\partial\Omega$ ,  $u_0 = u_0(x)$  is a given initial datum, and  $A = A(p) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the *diffusion flux* (i.e., the negative of the heat flux density) of the evolution process. Here,  $u = u(x, t)$  is the density of some quantity at position  $x$  and time  $t$ , with  $Du = (u_{x_1}, \dots, u_{x_n})$  and  $u_t$  denoting its spatial gradient and rate of change, respectively.

In general, a function  $u \in W^{1,\infty}(\Omega_T)$  is called a *Lipschitz solution* to problem (1.1) provided that equality

$$\int_{\Omega} (u(x, s)\zeta(x, s) - u_0(x)\zeta(x, 0))dx = \int_0^s \int_{\Omega} (u\zeta_t - A(Du) \cdot D\zeta)dxdt$$

holds for each  $\zeta \in C^\infty(\bar{\Omega}_T)$  and each  $s \in [0, T]$ . Let  $\zeta \equiv 1$ ; then it is immediate from the definition that any Lipschitz solution  $u$  to (1.1) conserves the total quantity over time:

$$\int_{\Omega} u(x, t)dx = \int_{\Omega} u_0(x)dx \quad \forall t \in [0, T].$$

The usual evolution of heat equation corresponds to the case of *isotropic* diffusion given by Fourier's law:  $A(p) = kp$  ( $p \in \mathbb{R}^n$ ), where  $k > 0$  is the

---

2010 *Mathematics Subject Classification.* 35M13, 35K20, 35D30, 49K20.

*Key words and phrases.* forward-backward parabolic equations, partial differential inclusions, convex integration, Baire's category method, infinitely many Lipschitz solutions.

thermal diffusivity. More generally, for standard diffusions, the diffusion flux  $A(p)$  is assumed to be *monotone*; namely,

$$(1.2) \quad (A(p) - A(q)) \cdot (p - q) \geq 0 \quad (p, q \in \mathbb{R}^n).$$

In this case, problem (1.1) is parabolic and can be studied by the standard methods of parabolic equations, monotone operators and non-linear semi-group theory [10, 3, 11]. In particular, when  $A(p)$  is given by a smooth convex function  $W(p)$  through  $A(p) = D_p W(p)$  ( $p \in \mathbb{R}^n$ ), the diffusion equation in (1.1) can be viewed and thus studied as a certain gradient flow generated by the energy functional

$$I(u) = \int_{\Omega} W(Du(x)) dx.$$

On the other hand, for certain applications of the evolution process to some important physical problems, the underlying diffusion flux  $A(p)$  may be *non-monotone*, yielding non-parabolic problem (1.1). In this regard, our recent paper [9] studied the existence and properties of Lipschitz solutions to (1.1) for some non-monotone diffusion fluxes  $A(p)$  of the form

$$(1.3) \quad A(p) = f(|p|^2)p \quad (p \in \mathbb{R}^n),$$

given by a function  $f: [0, \infty) \rightarrow \mathbb{R}$  with *profile*  $\sigma(s) = sf(s^2)$  having one of the graphs in Figures 1 and 2, referred to as the *Perona-Malik type* in image processing and the *Höllig type* related to the phase transitions in thermodynamics, respectively; for more details, see [9] and the references therein.

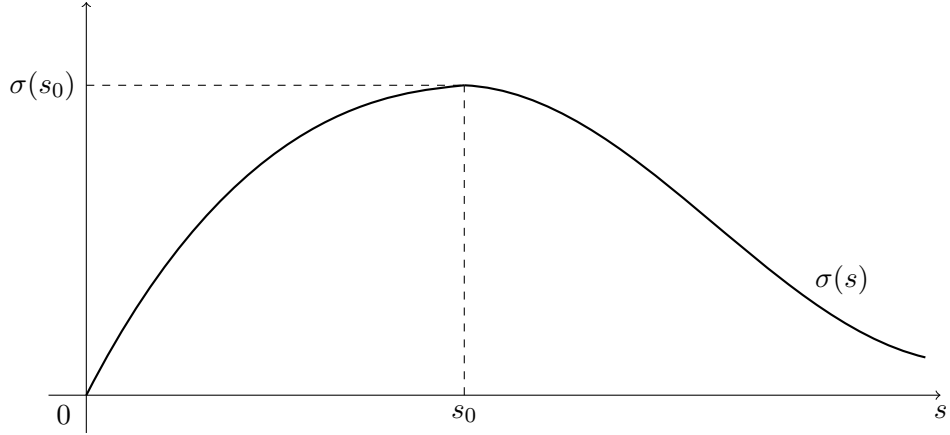
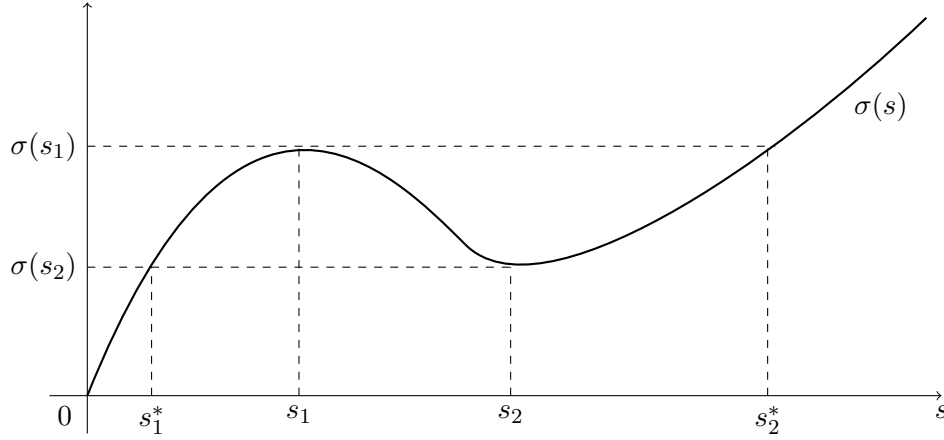


FIGURE 1. Perona-Malik type profiles  $\sigma(s)$ .

Parabolic and non-parabolic problems (1.1) discussed so far have used diffusion fluxes  $A(p)$  fulfilling *Fourier's inequality*:

$$(1.4) \quad A(p) \cdot p \geq 0 \quad (p \in \mathbb{R}^n),$$

FIGURE 2. Höllig type profiles  $\sigma(s)$ .

which is consistent to the Clausius-Duhem inequality in the second law of thermodynamics; see, e.g. [6, 13]. Observe that the inequality of monotonicity (1.2) implies Fourier's inequality (1.4), but the converse does not hold.

In this paper, we consider diffusion fluxes  $A(p)$  of the form (1.3) with profiles  $\sigma(s)$  having the graphs as in Figure 3; in this case, Fourier's inequality (1.4) is violated as

$$\begin{cases} A(p) \cdot p > 0, & |p| > s_0, \\ A(p) \cdot p < 0, & 0 < |p| < s_0, \\ A(p) \cdot p = 0, & |p| \in \{0, s_0\}. \end{cases}$$

We will call such profiles  $\sigma(s)$  as *non-Fourier type*. More precisely, we impose the following conditions on the non-Fourier type profiles  $\sigma(s) = sf(s^2)$ .

**Hypothesis (NF):** (See Figure 3.)

- (i) There exist two numbers  $s_0 > s_- > 0$  such that

$$f \in C^0([0, \infty)) \cap C^1((0, s_-^2) \cup (s_-^2, s_0^2)) \cap C^{1+\alpha}(s_0^2, \infty),$$

where  $0 < \alpha < 1$  is any fixed number.

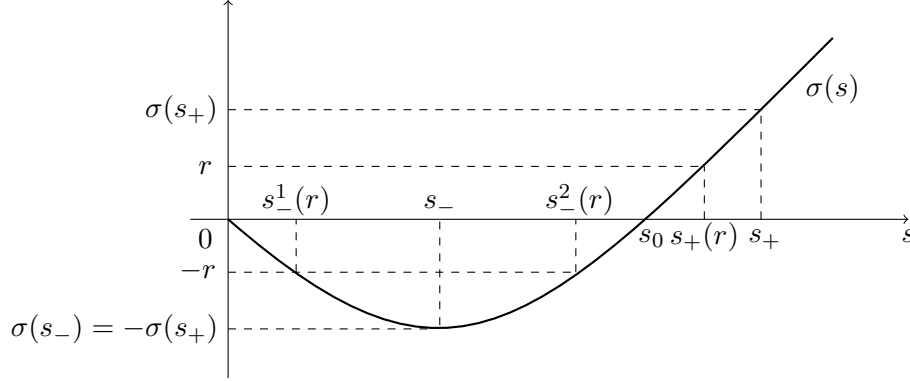
- (ii)  $\sigma(s_0) = 0$ ,  $\sigma'(s) < 0 \ \forall s \in (0, s_-)$ ,  $\sigma'(s) > 0 \ \forall s \in (s_-, s_0) \cup (s_0, \infty)$ , and  $\lambda \leq \sigma'(s) \leq \Lambda \ \forall s > 2s_0$ , where  $\Lambda \geq \lambda > 0$  are constants.

- (iii) Let  $s_+ \in (s_0, \infty)$  denote the unique number with  $\sigma(s_+) = -\sigma(s_-)$ .

Also, for each  $r \in (0, \sigma(s_+))$ , let  $s_+(r) \in (s_0, s_+)$ ,  $s_-^1(r) \in (0, s_-)$  and  $s_-^2(r) \in (s_-, s_0)$  denote the unique numbers such that

$$r = \sigma(s_+(r)) = -\sigma(s_-^1(r)) = -\sigma(s_-^2(r)).$$

The main purpose of this paper is to explore the scope of the methods of [8, 9] in the application to problem (1.1) in all dimensions for diffusion

FIGURE 3. Non-Fourier type profiles  $\sigma(s)$ .

profiles  $\sigma(s)$  of the non-Fourier type. To state our main theorem, we make the following assumptions on the domain  $\Omega$  and initial datum  $u_0$ :

$$(1.5) \quad \begin{cases} \Omega \subset \mathbb{R}^n \text{ is a bounded domain with } \partial\Omega \text{ of class } C^{2+\alpha}, \\ u_0 \in C^{2+\alpha}(\bar{\Omega}) \text{ is non-constant with } Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0. \end{cases}$$

In addition to this, we further assume without loss of generality that the initial datum  $u_0$  satisfies

$$(1.6) \quad \int_{\Omega} u_0(x) dx = 0,$$

since otherwise we may solve problem (1.1) with initial datum  $\tilde{u}_0 = u_0 - \bar{u}_0$ , where  $\bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0 dx$ .

We now state the main existence theorem as follows.

**Theorem 1.1.** *Let  $m_0 = \min_{\bar{\Omega}} |Du_0|$  and  $m'_0 = \max\{m_0, s_0\}$ , and assume  $|Du_0(x_0)| \in (0, s_+)$  at some  $x_0 \in \Omega$ . Then for each  $\tilde{r} \in (\sigma(m'_0), \sigma(s_+))$ , there exist an open set  $\Omega_T^{\tilde{r}} \subset \Omega_T$  and infinitely many Lipschitz solutions  $u$  to (1.1) of the following two types:*

**(Type I)**  $|Du| \in [s_-^2(\tilde{r}), s_+(\tilde{r})] \cup \{0\}$  a.e. in  $\Omega_T \setminus \Omega_T^{\tilde{r}}$ .

**(Type II)**  $|Du| \in [0, s_-^1(\tilde{r})] \cup [s_0, s_+(\tilde{r})]$  a.e. in  $\Omega_T \setminus \Omega_T^{\tilde{r}}$ .

**(Common for both types)**

$$u \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_T^{\tilde{r}}), \quad u_t = \operatorname{div}(A(Du)) \text{ pointwise in } \Omega_T^{\tilde{r}},$$

$$|Du(x, t)| > s_+(\tilde{r}) \quad \forall (x, t) \in \Omega_T^{\tilde{r}}, \quad \text{and} \quad \Omega_0^{\tilde{r}} \subset \partial\Omega_T^{\tilde{r}},$$

where  $\Omega_0^{\tilde{r}} = \{(x, 0) \mid x \in \Omega, |Du_0(x)| > s_+(\tilde{r})\}$ .

Lipschitz solutions  $u$  of **Type I** will be also called as *forward-forward type* or simply **FFT** solutions, and likewise for those  $u$  of **Type II** as *backward-forward type* or simply **BFT** solutions. For clear distinction of these two

types, we keep using boldface letters for **Type I**, **Type II**, **FFT** and **BFT** throughout the paper.

As a byproduct of Theorem 1.1, we have the simple existence theorem as follows.

**Theorem 1.2.** *Let  $\Omega$  be as in (1.5). Then for any initial datum  $u_0 \in C^{2+\alpha}(\bar{\Omega})$  with  $Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$ , problem (1.1) has at least one Lipschitz solution.*

The rest of the paper is organized as follows. Section 2 begins with a general density approach to problem (1.1) as a non-homogeneous partial differential inclusion. Then the general existence theorem, Theorem 2.1, is formulated under a key density hypothesis. Also, some essential ingredients for the proof of the main theorem, Theorem 1.1, are provided. As the pivotal analysis of the paper, the geometry of related matrix sets is investigated in Section 3, leading to the relaxation result on a homogeneous differential inclusion, Theorem 3.7. Section 4 is devoted to the simultaneous construction of suitable boundary functions and admissible sets for **Types I** and **II** in the stream of the proof of Theorem 1.1. Then the key density hypothesis for each type is realized in Section 5, completing the proof of Theorem 1.1. The proof of Theorem 1.2 is also included in the last part of this section. Lastly, Section 6 adds a remark on further existence results that can be deduced from the combination of [9] and this paper.

## 2. A GENERAL DENSITY APPROACH AND SOME USEFUL RESULTS

In this section, we present a general density method and some essential ingredients for the proof of the main result, Theorem 1.1. All the proofs and motivational ideas can be found in the previous paper [9] and references therein; so we do not repeat those here unless otherwise stated.

**2.1. Admissible set and the density approach.** For the general density approach to problem (1.1), we assume the following:

$$\begin{cases} \Omega \subset \mathbb{R}^n \text{ is a bounded Lipschitz domain,} \\ \text{the initial datum } u_0 \in W^{1,\infty}(\Omega), \\ \text{the diffusion flux } A \in C(\mathbb{R}^n; \mathbb{R}^n). \end{cases}$$

Assume that we have a function  $\Phi = (u^*, v^*) \in W^{1,\infty}(\Omega_T; \mathbb{R}^{1+n})$  satisfying

$$(2.1) \quad \begin{cases} u^*(x, 0) = u_0(x), & x \in \Omega, \\ \operatorname{div} v^*(x, t) = u^*(x, t), & \text{a.e. } (x, t) \in \Omega_T, \\ v^*(\cdot, t) \cdot \mathbf{n}|_{\partial\Omega} = 0, & t \in [0, T], \end{cases}$$

which will be called a *boundary function* for the initial datum  $u_0$ . We denote by  $W_{u^*}^{1,\infty}(\Omega_T)$ ,  $W_{v^*}^{1,\infty}(\Omega_T; \mathbb{R}^n)$  the usual *Dirichlet classes* with boundary traces  $u^*$ ,  $v^*$ , respectively.

We say that  $\mathcal{U} \subset W_{u^*}^{1,\infty}(\Omega_T)$  is an *admissible set* provided that it is nonempty and bounded in  $W_{u^*}^{1,\infty}(\Omega_T)$  and that for each  $u \in \mathcal{U}$ , there exists a vector function  $v \in W_{v^*}^{1,\infty}(\Omega_T; \mathbb{R}^n)$  satisfying

$$\operatorname{div} v = u \quad \text{a.e. in } \Omega_T \quad \text{and} \quad \|v_t\|_{L^\infty(\Omega_T)} \leq R,$$

where  $R > 0$  is any fixed number. If  $\mathcal{U}$  is an admissible set, for each  $\epsilon > 0$ , let  $\mathcal{U}_\epsilon$  be the set of all  $u \in \mathcal{U}$  such that there exists a function  $v \in W_{v^*}^{1,\infty}(\Omega_T; \mathbb{R}^n)$  satisfying

$$\begin{aligned} \operatorname{div} v &= u \quad \text{a.e. in } \Omega_T, \quad \|v_t\|_{L^\infty(\Omega_T)} \leq R, \quad \text{and} \\ \int_{\Omega_T} |v_t(x, t) - A(Du(x, t))| \, dx dt &\leq \epsilon |\Omega_T|. \end{aligned}$$

We now have the following general existence theorem under a pivotal density hypothesis of  $\mathcal{U}_\epsilon$  in  $\mathcal{U}$ . Although the proof of this theorem already appeared in [8, 9], we need to reproduce it here since the proof itself will be used in the proof of Theorem 1.1.

**Theorem 2.1.** *Let  $\mathcal{U} \subset W_{u^*}^{1,\infty}(\Omega_T)$  be an admissible set satisfying the density property:*

$$\mathcal{U}_\epsilon \text{ is dense in } \mathcal{U} \text{ under the } L^\infty\text{-norm for each } \epsilon > 0.$$

*Then, given any  $\varphi \in \mathcal{U}$ , for each  $\delta > 0$ , there exists a Lipschitz solution  $u \in W_{u^*}^{1,\infty}(\Omega_T)$  to (1.1) satisfying  $\|u - \varphi\|_{L^\infty(\Omega_T)} < \delta$ . Furthermore, if  $\mathcal{U}$  contains a function which is not a Lipschitz solution to (1.1), then (1.1) itself admits infinitely many Lipschitz solutions.*

*Proof.* For clarity, we divide the proof into several steps.

1. Let  $\mathcal{X}$  be the closure of  $\mathcal{U}$  in the metric space  $L^\infty(\Omega_T)$ . Then  $(\mathcal{X}, L^\infty)$  is a non-empty complete metric space. By assumption, each  $\mathcal{U}_\epsilon$  is dense in  $\mathcal{X}$ . Moreover, since  $\mathcal{U}$  is bounded in  $W_{u^*}^{1,\infty}(\Omega_T)$ , we have  $\mathcal{X} \subset W_{u^*}^{1,\infty}(\Omega_T)$ .

2. Let  $\mathcal{Y} = L^1(\Omega_T; \mathbb{R}^n)$ . For  $h > 0$ , define  $T_h: \mathcal{X} \rightarrow \mathcal{Y}$  as follows. Given any  $u \in \mathcal{X}$ , write  $u = u^* + w$  with  $w \in W_0^{1,\infty}(\Omega_T)$  and define

$$T_h(u) = Du^* + D(\rho_h * w),$$

where  $\rho_h(z) = h^{-N} \rho(z/h)$ , with  $z = (x, t)$  and  $N = n + 1$ , is the standard  $h$ -mollifier in  $\mathbb{R}^N$ , and  $\rho_h * w$  is the usual convolution in  $\mathbb{R}^N$  with  $w$  extended to be zero outside  $\bar{\Omega}_T$ . Then, for each  $h > 0$ , the map  $T_h: (\mathcal{X}, L^\infty) \rightarrow (\mathcal{Y}, L^1)$  is continuous, and for each  $u \in \mathcal{X}$ ,

$$\lim_{h \rightarrow 0^+} \|T_h(u) - Du\|_{L^1(\Omega_T)} = \lim_{h \rightarrow 0^+} \|\rho_h * Dw - Dw\|_{L^1(\Omega_T)} = 0.$$

Therefore, the spatial gradient operator  $D: \mathcal{X} \rightarrow \mathcal{Y}$  is the pointwise limit of a sequence of continuous maps  $T_h: \mathcal{X} \rightarrow \mathcal{Y}$ ; hence  $D: \mathcal{X} \rightarrow \mathcal{Y}$  is a *Baire-one map*. By Baire's category theorem (e.g., [4, Theorem 10.13]), there exists a *residual set*  $\mathcal{G} \subset \mathcal{X}$  such that the operator  $D$  is continuous at each point of  $\mathcal{G}$ . Since  $\mathcal{X} \setminus \mathcal{G}$  is of the *first category*, the set  $\mathcal{G}$  is *dense* in  $\mathcal{X}$ . Therefore,

given any  $\varphi \in \mathcal{X}$ , for each  $\delta > 0$ , there exists a function  $u \in \mathcal{G}$  such that  $\|u - \varphi\|_{L^\infty(\Omega_T)} < \delta$ .

3. We now prove that each  $u \in \mathcal{G}$  is a Lipschitz solution to (1.1). Let  $u \in \mathcal{G}$  be given. By the density of  $\mathcal{U}_\epsilon$  in  $(\mathcal{X}, L^\infty)$  for each  $\epsilon > 0$ , for every  $j \in \mathbb{N}$ , there exists a function  $u_j \in \mathcal{U}_{1/j}$  such that  $\|u_j - u\|_{L^\infty(\Omega_T)} < 1/j$ . Since the operator  $D: (\mathcal{X}, L^\infty) \rightarrow (\mathcal{Y}, L^1)$  is continuous at  $u$ , we have  $Du_j \rightarrow Du$  in  $L^1(\Omega_T; \mathbb{R}^n)$ . Furthermore, from (2.1) and the definition of  $\mathcal{U}_{1/j}$ , there exists a function  $v_j \in W_{v^*}^{1,\infty}(\Omega_T; \mathbb{R}^n)$  such that for each  $\zeta \in C^\infty(\bar{\Omega}_T)$  and each  $t \in [0, T]$ ,

$$(2.2) \quad \begin{aligned} \int_{\Omega} v_j(x, t) \cdot D\zeta(x, t) dx &= - \int_{\Omega} u_j(x, t) \zeta(x, t) dx, \\ \|(v_j)_t\|_{L^\infty(\Omega_T)} &\leq R, \quad \int_{\Omega_T} |(v_j)_t - A(Du_j)| dx dt \leq \frac{1}{j} |\Omega_T|. \end{aligned}$$

Since  $v_j(x, 0) = v^*(x, 0) \in W^{1,\infty}(\Omega; \mathbb{R}^n)$  and  $\|(v_j)_t\|_{L^\infty(\Omega_T)} \leq R$ , it follows that both sequences  $\{v_j\}$  and  $\{(v_j)_t\}$  are bounded in  $L^2(\Omega_T; \mathbb{R}^n) \approx L^2((0, T); L^2(\Omega; \mathbb{R}^n))$ . So we may assume

$$v_j \rightharpoonup v \text{ and } (v_j)_t \rightharpoonup v_t \text{ in } L^2((0, T); L^2(\Omega; \mathbb{R}^n))$$

for some  $v \in W^{1,2}((0, T); L^2(\Omega; \mathbb{R}^n))$ , where  $\rightharpoonup$  denotes the weak convergence. Upon taking the limit as  $j \rightarrow \infty$  in (2.2), since  $v \in C([0, T]; L^2(\Omega; \mathbb{R}^n))$  and  $A \in C(\mathbb{R}^n; \mathbb{R}^n)$ , we obtain

$$\begin{aligned} \int_{\Omega} v(x, t) \cdot D\zeta(x, t) dx &= - \int_{\Omega} u(x, t) \zeta(x, t) dx \quad (t \in [0, T]), \\ v_t(x, t) &= A(Du(x, t)) \quad a.e. (x, t) \in \Omega_T. \end{aligned}$$

Consequently, by [8, Lemma 3.1],  $u$  is a Lipschitz solution to (1.1).

4. Finally, assume  $\mathcal{U}$  contains a function which is not a Lipschitz solution to (1.1); hence  $\mathcal{G} \neq \mathcal{U}$ . Then  $\mathcal{G}$  cannot be a finite set, since otherwise the  $L^\infty$ -closure  $\mathcal{X} = \overline{\mathcal{G}} = \overline{\mathcal{U}}$  would be a finite set, making  $\mathcal{U} = \mathcal{G}$ . Therefore, in this case, (1.1) admits infinitely many Lipschitz solutions. The proof is complete.  $\square$

**2.2. Uniformly parabolic equations.** We refer to the standard references (e.g., [10, 11]) for some notations concerning functions and domains of class  $C^{k+\alpha}$  with an integer  $k \geq 0$ .

Assume  $\tilde{f} \in C^{1+\alpha}([0, \infty))$  is a function satisfying

$$(2.3) \quad \theta \leq \tilde{f}(s) + 2s\tilde{f}'(s) \leq \Theta \quad \forall s \geq 0,$$

where  $\Theta \geq \theta > 0$  are constants. This condition is equivalent to  $\theta \leq (s\tilde{f}(s^2))' \leq \Theta$  for all  $s \in \mathbb{R}$ ; hence,  $\theta \leq \tilde{f}(s) \leq \Theta$  for all  $s \geq 0$ . Let

$$\tilde{A}(p) = \tilde{f}(|p|^2)p \quad (p \in \mathbb{R}^n).$$

Then we have

$$\tilde{A}_{p_j}^i(p) = \tilde{f}(|p|^2)\delta_{ij} + 2\tilde{f}'(|p|^2)p_i p_j \quad (i, j = 1, 2, \dots, n; p \in \mathbb{R}^n)$$

and hence the *uniform ellipticity condition*:

$$\theta|q|^2 \leq \sum_{i,j=1}^n \tilde{A}_{p_j}^i(p) q_i q_j \leq \Theta|q|^2 \quad \forall p, q \in \mathbb{R}^n.$$

The proof of the following classical result can be found in [11, Theorem 13.24].

**Theorem 2.2.** *Assume (1.5). Then the initial-Neumann boundary value problem*

$$(2.4) \quad \begin{cases} u_t = \operatorname{div}(\tilde{A}(Du)) & \text{in } \Omega_T, \\ \partial u / \partial \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{for } x \in \Omega \end{cases}$$

*admits a unique solution*  $u \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_T)$ .

**2.3. Modified profile.** The following elementary result can be proved in a similar way as in [14]; we omit the proof.

**Lemma 2.3** (see Figure 4). *Assume Hypothesis (NF). Then for each  $0 < r < \sigma(s_+)$ , there exists a function  $\tilde{\sigma} \in C^{1+\alpha}([0, \infty))$  with  $\tilde{\sigma}(0) = 0$  such that  $\tilde{\sigma}$  is linear near 0 and that*

$$\begin{cases} \tilde{\sigma}(s) > \sigma(s), & 0 < s < s_+(r), \\ \tilde{\sigma}(s) = \sigma(s), & s_+(r) \leq s < \infty, \\ \theta \leq \tilde{\sigma}'(s) \leq \Theta, & 0 \leq s < \infty \end{cases}$$

*for some constants  $\Theta \geq \theta > 0$ . With such a function  $\tilde{\sigma}$ , define  $\tilde{f}(s) = \tilde{\sigma}(\sqrt{s})/\sqrt{s}$  ( $s > 0$ ) and  $\tilde{f}(0) = \lim_{s \rightarrow 0^+} \tilde{\sigma}(\sqrt{s})/\sqrt{s}$ ; then  $\tilde{f} \in C^{1+\alpha}([0, \infty))$  fulfills condition (2.3).*

**2.4. Right inverse of the divergence operator.** We follow an argument of Bourgain and Brezis [2, Lemma 4] to construct a right inverse  $\mathcal{R}$  of the divergence operator:  $\operatorname{div} \mathcal{R} = \operatorname{Id}$  (in the sense of distributions in  $\Omega_T$ ). For the purpose of this paper, the construction of  $\mathcal{R}$  is restricted to a *box*, by which we mean a domain  $Q$  given by  $Q = J_1 \times J_2 \times \cdots \times J_n$ , where  $J_i = (a_i, b_i) \subset \mathbb{R}$  is a finite open interval.

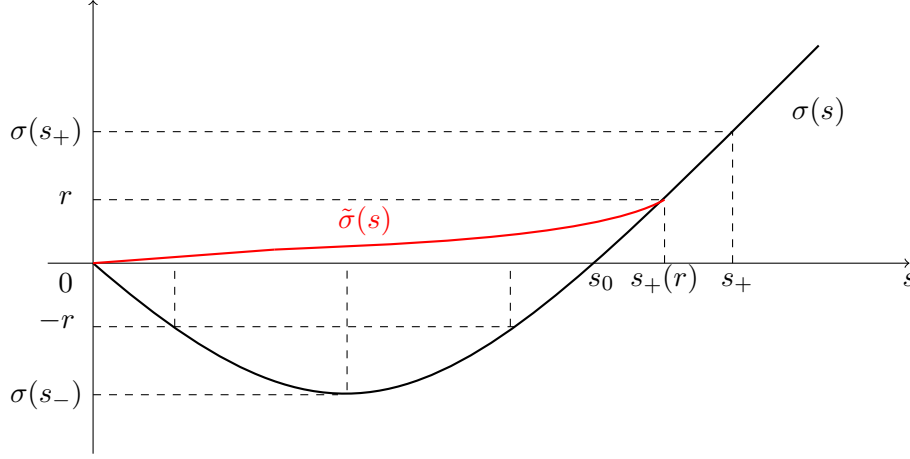
We have the following result [8, Theorem 2.3].

**Theorem 2.4.** *Let  $Q \times I$  be a box in  $\mathbb{R}^{n+1}$ , where  $I \subset \mathbb{R}$  is a finite open interval. Then there is a bounded linear operator  $\mathcal{R} = \mathcal{R}_n: L^\infty(Q \times I) \rightarrow L^\infty(Q \times I; \mathbb{R}^n)$  satisfying the following: If  $u \in W_0^{1,\infty}(Q \times I)$  is such that  $\int_Q u(x, t) dx = 0$  for all  $t \in I$ , then  $v := \mathcal{R}u \in W_0^{1,\infty}(Q \times I; \mathbb{R}^n)$ ,  $\operatorname{div} v = u$  a.e. in  $Q \times I$ , and*

$$(2.5) \quad \|v_t\|_{L^\infty(Q \times I)} \leq C_n (|J_1| + \cdots + |J_n|) \|u_t\|_{L^\infty(Q \times I)},$$

*where  $Q = J_1 \times \cdots \times J_n$  and  $C_n > 0$  is a dimensional constant. Moreover, if  $u \in C^1(\overline{Q \times I})$ , then  $v \in C^1(\overline{Q \times I}; \mathbb{R}^n)$ .*



FIGURE 4. Non-Fourier type profile  $\sigma(s)$  and modified  $\tilde{\sigma}(s)$ .

### 3. GEOMETRY OF THE RELEVANT MATRIX SETS

Let  $A(p)$  be the diffusion flux given by (1.3) with profile  $\sigma(s) = sf(s^2)$  satisfying Hypothesis (NF). Let  $K_0$  be the subset of the  $(1+n) \times (n+1)$  matrix space  $\mathbb{M}^{(1+n) \times (n+1)}$  defined by

$$K_0 = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \mid p \in \mathbb{R}^n, c \in \mathbb{R}, B \in \mathbb{M}^{n \times n}, \text{tr } B = 0 \right\}.$$

Under Hypothesis (NF), certain structures of the set  $K_0$  turn out to be quite useful, especially when it comes to the relaxation of homogeneous partial differential inclusion  $\nabla \omega(z) \in K_0$  with  $z = (x, t)$  and  $\omega = (\varphi, \psi)$ . We investigate these structures and establish such a relaxation result for both **FFT** and **BFT** solutions throughout this section.

**3.1. Geometry of the matrix set  $K_0$ .** We study some subsets of  $K_0$ , depending on the different types of solutions to be sought.

**Type I: FFT solutions.** In search for this type of solutions, we adopt the following notations. Fix any two numbers  $0 < r_1 < r_2 < \sigma(s_+)$ , and let  $F_0 = F_{0,r_1,r_2}$  be the subset of  $K_0$  defined by

$$F_0 = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \mid \begin{array}{l} p \in \mathbb{R}^n, |p| \in (s_-^2(r_2), s_-^2(r_1)) \cup (s_+(r_1), s_+(r_2)), \\ c \in \mathbb{R}, B \in \mathbb{M}^{n \times n}, \text{tr } B = 0 \end{array} \right\}.$$

We decompose the set  $F_0$  into two disjoint subsets as follows:

$$\begin{aligned} F_- &= F_{-,r_1,r_2} = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \mid \begin{array}{l} p \in \mathbb{R}^n, |p| \in (s_-^2(r_2), s_-^2(r_1)), \\ c \in \mathbb{R}, B \in \mathbb{M}^{n \times n}, \text{tr } B = 0 \end{array} \right\}, \\ F_+ &= F_{+,r_1,r_2} = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \mid \begin{array}{l} p \in \mathbb{R}^n, |p| \in (s_+(r_1), s_+(r_2)), \\ c \in \mathbb{R}, B \in \mathbb{M}^{n \times n}, \text{tr } B = 0 \end{array} \right\}. \end{aligned}$$

**Type II: BFT solutions.** To handle this type of solutions, we use the following notations. Fix any two numbers  $0 < r_1 < r_2 < \sigma(s_+)$ , and let  $F_0 = F_{0,r_1,r_2}$  be the subset of  $K_0$  given by

$$F_0 = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \mid \begin{array}{l} p \in \mathbb{R}^n, |p| \in (s_-^1(r_1), s_-^1(r_2)) \cup (s_+(r_1), s_+(r_2)), \\ c \in \mathbb{R}, B \in \mathbb{M}^{n \times n}, \text{tr } B = 0 \end{array} \right\}.$$

The set  $F_0$  is then decomposed into two disjoint subsets as follows:

$$F_- = F_{-,r_1,r_2} = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \mid \begin{array}{l} p \in \mathbb{R}^n, |p| \in (s_-^1(r_1), s_-^1(r_2)), \\ c \in \mathbb{R}, B \in \mathbb{M}^{n \times n}, \text{tr } B = 0 \end{array} \right\},$$

$$F_+ = F_{+,r_1,r_2} = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \mid \begin{array}{l} p \in \mathbb{R}^n, |p| \in (s_+(r_1), s_+(r_2)), \\ c \in \mathbb{R}, B \in \mathbb{M}^{n \times n}, \text{tr } B = 0 \end{array} \right\}.$$

In order to study the homogeneous differential inclusion  $\nabla \omega(z) \in K_0$ , we first scrutinize the rank-one structure of the set  $F_0 \subset K_0$  for each type. To this aim, we define

$$R(F_0) = \bigcup_{\xi_{\pm} \in F_{\pm}, \text{rank}(\xi_+ - \xi_-)=1} (\xi_-, \xi_+),$$

where  $(\xi_-, \xi_+)$  is the open line segment in  $\mathbb{M}^{(1+n) \times (n+1)}$  joining  $\xi_{\pm}$ . We now explore the structure of the set  $R(F_0)$  in detail for both types simultaneously.

**3.1.1. Alternate expression for  $R(F_0)$ .** We establish more specific criteria for matrices in  $R(F_0)$  than its definition.

**Lemma 3.1.** *Let  $\xi \in \mathbb{M}^{(1+n) \times (n+1)}$ . Then  $\xi \in R(F_0)$  if and only if there exist numbers  $t_- < 0 < t_+$  and vectors  $q, \gamma \in \mathbb{R}^n$  with  $|q| = 1, \gamma \cdot q = 0$  such that for each  $b \in \mathbb{R} \setminus \{0\}$ , if  $\eta = \begin{pmatrix} q & b \\ \frac{1}{b}q \otimes \gamma & \gamma \end{pmatrix}$ , then  $\xi + t_{\pm}\eta \in F_{\pm}$ .*

*Proof.* Assume  $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0)$ . By definition,  $\xi + t_{\pm}\tilde{\eta} \in F_{\pm}$ , where  $t_- < 0 < t_+$  and  $\tilde{\eta}$  is a rank-one matrix given by

$$\tilde{\eta} = \begin{pmatrix} a \\ \alpha \end{pmatrix} \otimes (q, \tilde{b}) = \begin{pmatrix} aq & a\tilde{b} \\ \alpha \otimes q & \tilde{b}\alpha \end{pmatrix}, \quad a^2 + |\alpha|^2 \neq 0, \quad \tilde{b}^2 + |q|^2 \neq 0,$$

for some  $a, \tilde{b} \in \mathbb{R}$  and  $\alpha, q \in \mathbb{R}^n$ ; here  $\alpha \otimes q$  denotes the rank-one or zero matrix  $(\alpha_i q_j)$  in  $\mathbb{M}^{n \times n}$ . Condition  $\xi + t_{\pm}\tilde{\eta} \in F_{\pm}$  with  $t_- < 0 < t_+$  is equivalent to the following: For **Type I**,

$$(3.1) \quad \begin{aligned} \text{tr } B = 0, \quad \alpha \cdot q = 0, \quad A(p + t_{\pm}aq) = \beta + t_{\pm}\tilde{b}\alpha, \\ |p + t_+aq| \in (s_+(r_1), s_+(r_2)), \quad |p + t_-aq| \in (s_-^2(r_2), s_-^2(r_1)). \end{aligned}$$

For **Type II**,

$$(3.2) \quad \begin{aligned} \text{tr } B = 0, \quad \alpha \cdot q = 0, \quad A(p + t_{\pm}aq) = \beta + t_{\pm}\tilde{b}\alpha, \\ |p + t_+aq| \in (s_+(r_1), s_+(r_2)), \quad |p + t_-aq| \in (s_-^1(r_1), s_-^1(r_2)). \end{aligned}$$

Therefore,  $aq \neq 0$ . Upon rescaling  $\tilde{\eta}$  and  $t_{\pm}$ , we can assume  $a = 1$  and  $|q| = 1$ ; namely,

$$\tilde{\eta} = \begin{pmatrix} q & \tilde{b} \\ \alpha \otimes q & \tilde{b}\alpha \end{pmatrix}, \quad |q| = 1, \quad \alpha \cdot q = 0.$$

We now set  $\gamma = \tilde{b}\alpha$ . Let  $b \in \mathbb{R} \setminus \{0\}$  and

$$\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}.$$

From (3.1) (**Type I**), (3.2) (**Type II**), it follows that  $\xi + t_{\pm}\eta \in F_{\pm}$ .

The converse easily follows from the definition of  $R(F_0)$ .  $\square$

**3.1.2. Diagonal components of matrices in  $R(F_0)$ .** The following gives a description for the diagonal components of matrices in  $R(F_0)$  for both types; the proof is precisely the same as that of [9, Lemma 5.3] and thus not reproduced here.

**Lemma 3.2.**

$$(3.3) \quad R(F_0) = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \mid c \in \mathbb{R}, B \in \mathbb{M}^{n \times n}, \operatorname{tr} B = 0, (p, \beta) \in \mathcal{S} \right\}$$

for some set  $\mathcal{S} = \mathcal{S}_{r_1, r_2} \subset \mathbb{R}^{n+n}$ .

**3.1.3. Selection of approximate collinear rank-one connections for  $R(F_0)$ .** We first give a 2-dimensional description for the rank-one connections of diagonal components of matrices in  $R(F_0)$  in a general form. The following lemma is common for both **Types I** and **II**.

**Lemma 3.3.** *For all positive numbers  $a, b$  and  $c$  with  $b > a$ , there exists a nonnegative continuous function  $h(a, b, c; \cdot, \cdot, \cdot, \cdot, \cdot, \cdot)$  defined on*

$$I_{a,b,c} := [0, a) \times \left[0, \frac{b-a}{2}\right) \times \left[0, \frac{b-a}{2}\right) \times [0, \infty) \times [0, c) \times [0, \infty)$$

with  $h(a, b, c; 0, 0, 0, 0, 0, 0) = 0$  satisfying the following:

Let  $\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1$  and  $\eta_2$  be any positive numbers with

$$0 < a - \delta_{11} < a < a + \delta_{12} < \frac{a+b}{2} < b - \delta_{21}, \quad 0 < c - \eta_1,$$

and let  $R_1 \in [a - \delta_{11}, a + \delta_{12}]$ ,  $R_2 \in [b - \delta_{21}, b + \delta_{22}]$ , and  $\tilde{R}_1, \tilde{R}_2 \in [c - \eta_1, c + \eta_2]$ . Suppose  $\theta \in [-\pi/2, \pi/2]$  and

$$\begin{aligned} & \left( \tilde{R}_1 \left( \cos\left(\frac{\pi}{2} + \theta\right), \sin\left(\frac{\pi}{2} + \theta\right) \right) - \tilde{R}_2 \left( \cos\left(\frac{\pi}{2} - \theta\right), \sin\left(\frac{\pi}{2} - \theta\right) \right) \right) \\ & \cdot \left( R_1 \left( \cos\left(\theta - \frac{\pi}{2}\right), \sin\left(\theta - \frac{\pi}{2}\right) \right) - R_2 \left( \cos\left(\frac{\pi}{2} - \theta\right), \sin\left(\frac{\pi}{2} - \theta\right) \right) \right) = 0. \end{aligned}$$

Then  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ ,  $\tilde{R}_1 \geq \tilde{R}_2$ , and

$$\max \left\{ \left| (0, -a) - R_1 \left( \cos\left(\theta - \frac{\pi}{2}\right), \sin\left(\theta - \frac{\pi}{2}\right) \right) \right|, \left| (0, b) - R_2 \left( \cos\left(\frac{\pi}{2} - \theta\right), \sin\left(\frac{\pi}{2} - \theta\right) \right) \right| \right\},$$

$$\begin{aligned} & \left| (0, c) - \tilde{R}_1 \left( \cos\left(\frac{\pi}{2} + \theta\right), \sin\left(\frac{\pi}{2} + \theta\right) \right) \right|, \left| (0, c) - \tilde{R}_2 \left( \cos\left(\frac{\pi}{2} - \theta\right), \sin\left(\frac{\pi}{2} - \theta\right) \right) \right| \Big\} \\ & \leq h(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2). \end{aligned}$$

*Proof.* By assumption,

$$\begin{aligned} 0 &= (\tilde{R}_1(-\sin \theta, \cos \theta) - \tilde{R}_2(\sin \theta, \cos \theta)) \cdot (R_1(\sin \theta, -\cos \theta) - R_2(\sin \theta, \cos \theta)) \\ &= (-(\tilde{R}_1 + \tilde{R}_2) \sin \theta, (\tilde{R}_1 - \tilde{R}_2) \cos \theta) \cdot ((R_1 - R_2) \sin \theta, -(R_1 + R_2) \cos \theta) \\ &= (\tilde{R}_1 + \tilde{R}_2)(R_2 - R_1) \sin^2 \theta - (\tilde{R}_1 - \tilde{R}_2)(R_1 + R_2) \cos^2 \theta, \end{aligned}$$

that is,

$$(\tilde{R}_1 - \tilde{R}_2)(R_1 + R_2) \cos^2 \theta = (\tilde{R}_1 + \tilde{R}_2)(R_2 - R_1) \sin^2 \theta;$$

hence,  $\theta \neq \pm \frac{\pi}{2}$ ,  $\tilde{R}_1 \geq \tilde{R}_2$ , and

$$\theta = \pm \tan^{-1} \left( \sqrt{\frac{(\tilde{R}_1 - \tilde{R}_2)(R_1 + R_2)}{(\tilde{R}_1 + \tilde{R}_2)(R_2 - R_1)}} \right).$$

So

$$\begin{aligned} |\theta| &\leq \tan^{-1} \left( \sqrt{\frac{(a + b + \delta_{12} + \delta_{22})(\eta_1 + \eta_2)}{2(b - a - \delta_{12} - \delta_{21})(c - \eta_1)}} \right) \\ &=: g(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2). \end{aligned}$$

Note that the function  $g(a, b, c; \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) : I_{a,b,c} \rightarrow [0, \pi/2)$  is well-defined and continuous and that  $g(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2) = 0$  whenever

$$(\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2) \in I_{a,b,c}, \quad \eta_1 = \eta_2 = 0.$$

Observe now that

$$\begin{aligned} & |(0, -a) - R_1(\cos(\theta - \frac{\pi}{2}), \sin(\theta - \frac{\pi}{2}))| \\ & \leq \max\{|(0, -a) - (a + \delta_{12})(\sin \theta, -\cos \theta)|, |(0, -a) - (a - \delta_{11})(\sin \theta, -\cos \theta)|\} \\ & = \max \left\{ \sqrt{(a + \delta_{12})^2 \sin^2 \theta + (a - (a + \delta_{12}) \cos \theta)^2}, \right. \\ & \quad \left. \sqrt{(a - \delta_{11})^2 \sin^2 \theta + (a - (a - \delta_{11}) \cos \theta)^2} \right\} \\ & = \max \left\{ \sqrt{(a + \delta_{12})^2 + a^2 - 2a(a + \delta_{12}) \cos \theta}, \right. \\ & \quad \left. \sqrt{(a - \delta_{11})^2 + a^2 - 2a(a - \delta_{11}) \cos \theta} \right\} \\ & \leq \max \left\{ \sqrt{(a + \delta_{12})^2 + a^2 - 2a(a + \delta_{12}) \cos(g(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2))}, \right. \\ & \quad \left. \sqrt{(a - \delta_{11})^2 + a^2 - 2a(a - \delta_{11}) \cos(g(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2))} \right\} \\ & =: h_1(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2), \\ & |(0, b) - R_2(\cos(\frac{\pi}{2} - \theta), \sin(\frac{\pi}{2} - \theta))| \\ & \leq \max\{|(0, b) - (b + \delta_{22})(\sin \theta, \cos \theta)|, |(0, b) - (b - \delta_{21})(\sin \theta, \cos \theta)|\} \end{aligned}$$

$$\begin{aligned}
&= \max \left\{ \sqrt{(b + \delta_{22})^2 \sin^2 \theta + (b - (b + \delta_{22}) \cos \theta)^2}, \right. \\
&\quad \left. \sqrt{(b - \delta_{21})^2 \sin^2 \theta + (b - (b - \delta_{21}) \cos \theta)^2} \right\} \\
&= \max \left\{ \sqrt{(b + \delta_{22})^2 + b^2 - 2b(b + \delta_{22}) \cos \theta}, \right. \\
&\quad \left. \sqrt{(b - \delta_{21})^2 + b^2 - 2b(b - \delta_{21}) \cos \theta} \right\} \\
&\leq \max \left\{ \sqrt{(b + \delta_{22})^2 + b^2 - 2b(b + \delta_{22}) \cos(g(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2))}, \right. \\
&\quad \left. \sqrt{(b - \delta_{21})^2 + b^2 - 2b(b - \delta_{21}) \cos(g(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2))} \right\} \\
&=: h_2(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2), \\
&\quad |(0, c) - \tilde{R}_1(\cos(\frac{\pi}{2} + \theta), \sin(\frac{\pi}{2} + \theta))| \\
&\leq \max \left\{ \sqrt{(c + \eta_2)^2 + c^2 - 2c(c + \eta_2) \cos(g(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2))}, \right. \\
&\quad \left. \sqrt{(c - \eta_1)^2 + c^2 - 2c(c - \eta_1) \cos(g(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2))} \right\} \\
&=: h_3(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2), \\
&\quad |(0, c) - \tilde{R}_2(\cos(\frac{\pi}{2} - \theta), \sin(\frac{\pi}{2} - \theta))| \leq h_3(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2).
\end{aligned}$$

Define

$$h(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2) = \max_{1 \leq j \leq 3} h_j(a, b, c; \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \eta_1, \eta_2).$$

Then it is easy to see that the function  $h(a, b, c; \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) : I_{a,b,c} \rightarrow [0, \infty)$  is well-defined and satisfies the required properties.  $\square$

We now apply the previous lemma to choose *approximate* collinear rank-one connections for the diagonal components of matrices in  $R(F_0)$ .

**Theorem 3.4.** *For each  $0 < r < \sigma(s_+)$ , there exists a number  $\mu_r > 0$  with  $0 < r - \mu_r < r + \mu_r < \sigma(s_+)$  satisfying the following:*

*Let  $0 < \mu \leq \mu_r$ , and let  $p_{\pm} \in \mathbb{R}^n$  satisfy*

$$s_-^2(r + \mu) < |p_-| < s_-^2(r - \mu) < s_+(r - \mu) < |p_+| < s_+(r + \mu), \quad (\textbf{Type I})$$

$$s_-^1(r - \mu) < |p_-| < s_-^1(r + \mu) < s_+(r - \mu) < |p_+| < s_+(r + \mu) \quad (\textbf{Type II})$$

*and  $(A(p_+) - A(p_-)) \cdot (p_+ - p_-) = 0$ . Then there exists a vector  $\zeta^0 \in \mathbb{S}^{n-1}$  such that, with  $p_+^0 = s_+(r)\zeta^0$ ,  $p_-^0 = -s_-^2(r)\zeta^0$  (**Type I**),  $p_-^0 = -s_-^1(r)\zeta^0$  (**Type II**),  $A(p_{\pm}^0) = r\zeta^0$ , we have*

$$\begin{aligned}
&\max\{|p_-^0 - p_-|, |p_+^0 - p_+|, |A(p_-^0) - A(p_-)|, |A(p_+^0) - A(p_+)|\} \\
&\leq h(s_-^2(r), s_+(r), r; s_-^2(r) - s_-^2(r + \mu), s_-^2(r - \mu) - s_-^2(r), \\
&\quad s_+(r) - s_+(r - \mu), s_+(r + \mu) - s_+(r), \mu, \mu), \quad (\textbf{Type I})
\end{aligned}$$

$$\begin{aligned}
& \max\{|p_-^0 - p_-|, |p_+^0 - p_+|, |A(p_-^0) - A(p_-)|, |A(p_+^0) - A(p_+)|\} \\
& \leq h(s_-^1(r), s_+(r), r; s_-^1(r) - s_-^1(r - \mu), s_-^1(r + \mu) - s_-^1(r), \\
& \quad s_+(r) - s_+(r - \mu), s_+(r + \mu) - s_+(r), \mu, \mu), \quad (\textbf{Type II})
\end{aligned}$$

where  $h$  is the function in Lemma 3.3.

*Proof.* Fix any  $0 < r < \sigma(s_+)$ . Since

$$\lim_{\mu \rightarrow 0^+} s_-^2(r - \mu) = s_-^2(r) < \frac{s_-^2(r) + s_+(r)}{2} < s_+(r) = \lim_{\mu \rightarrow 0^+} s_+(r - \mu), \quad (\textbf{Type I})$$

$$\lim_{\mu \rightarrow 0^+} s_-^1(r + \mu) = s_-^1(r) < \frac{s_-^1(r) + s_+(r)}{2} < s_+(r) = \lim_{\mu \rightarrow 0^+} s_+(r - \mu), \quad (\textbf{Type II})$$

we can find a  $\mu_r > 0$  so small that  $0 < r - \mu_r < r + \mu_r < \sigma(s_+)$  and that for every  $0 < \mu \leq \mu_r$ , we have

$$s_-^2(r - \mu) < \frac{s_-^2(r) + s_+(r)}{2} < s_+(r - \mu), \quad (\textbf{Type I})$$

$$s_-^1(r + \mu) < \frac{s_-^1(r) + s_+(r)}{2} < s_+(r - \mu). \quad (\textbf{Type II})$$

Now, let  $0 < \mu \leq \mu_r$ , and let  $p_{\pm} \in \mathbb{R}^n$  satisfy the conditions in the statement of the theorem. Let  $\Sigma_2$  denote the 2-dimensional linear subspace of  $\mathbb{R}^n$  spanned by the two vectors  $p_{\pm}$ . (In case of collinear  $p_{\pm}$ , we choose  $\Sigma_2$  to be any 2-dimensional linear space in  $\mathbb{R}^n$  containing  $p_{\pm}$ .) From the orthogonality condition, we have

$$\sigma(|p_+|)|p_+| + \sigma(|p_-|)|p_-| - \left( \frac{\sigma(|p_+|)}{|p_+|} + \frac{\sigma(|p_-|)}{|p_-|} \right) (p_+ \cdot p_-) = 0.$$

If  $p_{\pm}$  were pointing in the same direction, we would have  $\sigma(|p_+|) = \sigma(|p_-|)$ , a contradiction. Thus we can set

$$\zeta^0 = \frac{\frac{p_+}{|p_+|} - \frac{p_-}{|p_-|}}{\left| \frac{p_+}{|p_+|} - \frac{p_-}{|p_-|} \right|} \in \mathbb{S}^{n-1} \cap \Sigma_2.$$

Since the vectors  $p_{\pm}$ ,  $A(p_{\pm})$  and  $\zeta^0$  all lie in  $\Sigma_2$ , we can recast the problem into the setting of the previous lemma via one of the two linear isomorphisms of  $\Sigma_2$  onto  $\mathbb{R}^2$  with correspondence  $\zeta^0 \leftrightarrow (0, 1) \in \mathbb{R}^2$ . Then the result follows with the choices below in applying Lemma 3.3:  $a = s_-^2(r)$  (**Type I**),  $a = s_-^1(r)$  (**Type II**),  $b = s_+(r)$ ,  $c = r$ ,  $\delta_{11} = s_-^2(r) - s_-^2(r + \mu)$  (**Type I**),  $\delta_{11} = s_-^1(r) - s_-^1(r - \mu)$  (**Type II**),  $\delta_{12} = s_-^2(r - \mu) - s_-^2(r)$  (**Type I**),  $\delta_{12} = s_-^1(r + \mu) - s_-^1(r)$  (**Type II**),  $\delta_{21} = s_+(r) - s_+(r - \mu)$ ,  $\delta_{22} = s_+(r + \mu) - s_+(r)$ ,  $\eta_1 = \eta_2 = \mu$ ,  $R_1 = |p_-|$ ,  $R_2 = |p_+|$ ,  $\tilde{R}_1 = -\sigma(|p_-|)$ ,  $\tilde{R}_2 = \sigma(|p_+|)$ , and  $\theta \in [0, \pi/2]$  is the half of the angle between  $p_+$  and  $-p_-$ .  $\square$

**3.1.4. Final characterization of  $R(F_0)$ .** We are now ready to establish the result concerning the essential structure of  $R(F_0)$  for both types. For our purpose, it is sufficient to stick only to the diagonal components of matrices in  $R(F_0)$ .

**Theorem 3.5.** *Let  $0 < r < \sigma(s_+)$ . Then there exists a number  $\mu'_r > 0$  with  $0 < r - \mu'_r < r + \mu'_r < \sigma(s_+)$  such that for any  $0 < \mu \leq \mu'_r$ , the set  $\mathcal{S} = \mathcal{S}_{r-\mu, r+\mu} \subset \mathbb{R}^{n+n}$  in (3.3) satisfies the following:*

- (i)  $\sup_{(p, \beta) \in \mathcal{S}} |p| \leq s_+(r + \mu) < s_+$  and  $\sup_{(p, \beta) \in \mathcal{S}} |\beta| \leq r + \mu < \sigma(s_+)$ ; hence  $\mathcal{S}$  is bounded.
- (ii)  $\mathcal{S}$  is open.
- (iii) For each  $(p_0, \beta_0) \in \mathcal{S}$ , there exist an open set  $\mathcal{V} \subset \subset \mathcal{S}$  containing  $(p_0, \beta_0)$  and  $C^1$  functions  $q : \bar{\mathcal{V}} \rightarrow \mathbb{S}^{n-1}$ ,  $\gamma : \bar{\mathcal{V}} \rightarrow \mathbb{R}^n$ ,  $t_{\pm} : \bar{\mathcal{V}} \rightarrow \mathbb{R}$  with  $\gamma \cdot q = 0$  and  $t_- < 0 < t_+$  on  $\bar{\mathcal{V}}$  such that for every  $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0) = R(F_{0, r-\mu, r+\mu})$  with  $(p, \beta) \in \bar{\mathcal{V}}$ , we have

$$\xi + t_{\pm} \eta \in F_{\pm} = F_{\pm, r-\mu, r+\mu},$$

where  $t_{\pm} = t_{\pm}(p, \beta)$ ,  $\eta = \begin{pmatrix} q(p, \beta) & b \\ \frac{1}{b} \gamma(p, \beta) \otimes q(p, \beta) & \gamma(p, \beta) \end{pmatrix}$ , and  $b \neq 0$  is arbitrary.

*Proof.* Fix any  $0 < r < \sigma(s_+)$ . First, we let  $\mu > 0$  be any number with  $0 < r - \mu < r + \mu < \sigma(s_+)$  and prove (i). Then we choose later an upper bound  $\mu'_r$  of  $\mu$  for the validity of (ii) and (iii) above.

We divide the proof into several steps.

1. To show that (i) holds, choose any  $(p, \beta) \in \mathcal{S}$ . By Lemma 3.2,  $\xi := \begin{pmatrix} p & 0 \\ O & \beta \end{pmatrix} \in R(F_0)$ , where  $O$  is the  $n \times n$  zero matrix. By the definition of  $R(F_0)$ , there exist two matrices  $\xi_{\pm} = \begin{pmatrix} p_{\pm} & c_{\pm} \\ B_{\pm} & \sigma(|p_{\pm}|) \frac{p_{\pm}}{|p_{\pm}|} \end{pmatrix} \in F_{\pm}$  and a number  $0 < \lambda < 1$  such that  $\xi = \lambda \xi_+ + (1 - \lambda) \xi_-$ . So

$$|p| = |\lambda p_+ + (1 - \lambda) p_-| \leq s_+(r + \mu),$$

$$|\beta| = \left| \lambda \sigma(|p_+|) \frac{p_+}{|p_+|} + (1 - \lambda) \sigma(|p_-|) \frac{p_-}{|p_-|} \right| \leq r + \mu;$$

hence,  $\sup_{(p, \beta) \in \mathcal{S}} |p| \leq s_+(r + \mu)$  and  $\sup_{(p, \beta) \in \mathcal{S}} |\beta| \leq r + \mu$ . Thus,  $\mathcal{S}$  is bounded, and (i) is proved.

2. We now turn to the remaining assertions that the set  $\mathcal{S} = \mathcal{S}_{r-\mu, r+\mu}$  fulfills (ii) and (iii) for all sufficiently small  $\mu > 0$ . In this step, we still assume  $\mu > 0$  is any fixed number with  $0 < r - \mu < r + \mu < \sigma(s_+)$ .

Let  $(p_0, \beta_0) \in \mathcal{S}$ . Since  $\xi_0 := \begin{pmatrix} p_0 & 0 \\ O & \beta_0 \end{pmatrix} \in R(F_0)$ , it follows from Lemma 3.1 that there exist numbers  $t_0 < 0 < s_0$  and vectors  $q_0, \gamma_0 \in \mathbb{R}^n$  with  $|q_0| = 1$ ,  $\gamma_0 \cdot q_0 = 0$  such that  $\xi_0 + t_0 \eta_0 \in F_-$  and  $\xi_0 + s_0 \eta_0 \in F_+$ , where

$\eta_0 = \begin{pmatrix} q_0 & b \\ \frac{1}{b}q_0 \otimes \gamma_0 & \gamma_0 \end{pmatrix}$  and  $b \neq 0$  is any fixed number. Let  $q'_0 = t_0 q_0 \neq 0$ ,  $\gamma'_0 = t_0 \gamma_0$ , and  $s'_0 = s_0/t_0 < 0$ ; then

$$(3.4) \quad \begin{cases} \gamma'_0 \cdot q'_0 = 0, & s_+(r - \mu) < |p_0 + s'_0 q'_0| < s_+(r + \mu), \\ s_-^2(r + \mu) < |p_0 + q'_0| < s_-^2(r - \mu), & \textbf{(Type I)} \\ s_-^1(r - \mu) < |p_0 + q'_0| < s_-^1(r + \mu), & \textbf{(Type II)} \\ \sigma(|p_0 + s'_0 q'_0|) \frac{p_0 + s'_0 q'_0}{|p_0 + s'_0 q'_0|} = \beta_0 + s'_0 \gamma'_0, \\ \sigma(|p_0 + q'_0|) \frac{p_0 + q'_0}{|p_0 + q'_0|} = \beta_0 + \gamma'_0. \end{cases}$$

Observe also that

(3.5)

$$s_0 - t_0 \geq |(p_0 + s_0 q_0)| - |(p_0 + t_0 q_0)| > s_+(r - \mu) - s_-^2(r - \mu), \quad \textbf{(Type I)}$$

$$s_0 - t_0 \geq |(p_0 + s_0 q_0)| - |(p_0 + t_0 q_0)| > s_+(r - \mu) - s_-^1(r + \mu). \quad \textbf{(Type II)}$$

Next, consider the function  $F$  defined by

$$F(\gamma', q', s'; p, \beta) = \begin{pmatrix} \sigma(|p + s' q'|) \frac{p + s' q'}{|p + s' q'|} - \beta - s' \gamma' \\ \sigma(|p + q'|) \frac{p + q'}{|p + q'|} - \beta - \gamma' \\ \gamma' \cdot q' \end{pmatrix} \in \mathbb{R}^{n+n+1}$$

for all  $\gamma', q', p, \beta \in \mathbb{R}^n$  and  $s' \in \mathbb{R}$  with  $s_+(r - \mu) < |p + s' q'| < s_+(r + \mu)$ ,  $s_-^2(r + \mu) < |p + q'| < s_-^2(r - \mu)$  (**Type I**),  $s_-^1(r - \mu) < |p + q'| < s_-^1(r + \mu)$  (**Type II**). Then  $F$  is  $C^1$  in the described open subset of  $\mathbb{R}^{n+n+1+n+n}$ , and the observation (3.4) yields that

$$F(\gamma'_0, q'_0, s'_0; p_0, \beta_0) = 0.$$

Suppose for the moment that the Jacobian matrix  $D_{(\gamma', q', s')} F$  is invertible at the point  $(\gamma'_0, q'_0, s'_0; p_0, \beta_0)$ ; then the Implicit Function Theorem implies the following: There exist a bounded domain  $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_{(p_0, \beta_0)} \subset \mathbb{R}^{n+n}$  containing  $(p_0, \beta_0)$  and  $C^1$  functions  $\tilde{\gamma}, \tilde{q} \in \mathbb{R}^n$ ,  $\tilde{s} \in \mathbb{R}$  of  $(p, \beta) \in \tilde{\mathcal{V}}$  such that

$$\tilde{\gamma}(p_0, \beta_0) = \gamma'_0, \quad \tilde{q}(p_0, \beta_0) = q'_0, \quad \tilde{s}(p_0, \beta_0) = s'_0$$

and that

$$\tilde{s}(p, \beta) < 0, \quad s_+(r - \mu) < |p + \tilde{s}(p, \beta) \tilde{q}(p, \beta)| < s_+(r + \mu),$$

$$s_-^2(r + \mu) < |p + \tilde{q}(p, \beta)| < s_-^2(r - \mu), \quad \textbf{(Type I)}$$

$$s_-^1(r - \mu) < |p + \tilde{q}(p, \beta)| < s_-^1(r + \mu), \quad \textbf{(Type II)}$$

$$F(\tilde{\gamma}(p, \beta), \tilde{q}(p, \beta), \tilde{s}(p, \beta); p, \beta) = 0 \quad \forall (p, \beta) \in \tilde{\mathcal{V}}.$$

Define functions

$$\gamma = -\frac{\tilde{\gamma}}{|\tilde{q}|}, \quad q = -\frac{\tilde{q}}{|\tilde{q}|}, \quad t_+ = -\tilde{s}|\tilde{q}|, \quad t_- = -|\tilde{q}| \quad \text{in } \tilde{\mathcal{V}};$$

then

$$s_+(r - \mu) < |p + t_+ q| < s_+(r + \mu),$$



$$s_-^2(r + \mu) < |p + t_- q| < s_-^2(r - \mu), \quad (\text{Type I})$$

$$s_-^1(r - \mu) < |p + t_- q| < s_-^1(r + \mu), \quad (\text{Type II})$$

$$\sigma(|p + t_{\pm} q|) \frac{p + t_{\pm} q}{|p + t_{\pm} q|} = \beta + t_{\pm} \gamma, \quad |q| = 1, \quad \gamma \cdot q = 0, \quad t_- < 0 < t_+,$$

where  $(p, \beta) \in \tilde{\mathcal{V}}$ ,  $\gamma = \gamma(p, \beta)$ ,  $q = q(p, \beta)$ , and  $t_{\pm} = t_{\pm}(p, \beta)$ .

Let  $(p, \beta) \in \tilde{\mathcal{V}}$ ,  $B \in \mathbb{M}^{n \times n}$ ,  $\text{tr } B = 0$ ,  $b, c \in \mathbb{R}$ ,  $b \neq 0$ ,  $q = q(p, \beta)$ ,  $\gamma = \gamma(p, \beta)$ ,  $t_{\pm} = t_{\pm}(p, \beta)$ ,  $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix}$ , and  $\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}$ . Then  $\xi_{\pm} := \xi + t_{\pm} \eta \in F_{\pm}$ . By the definition of  $R(F_0)$ ,  $\xi \in (\xi_-, \xi_+) \subset R(F_0)$ . By Lemma 3.2, we thus have  $(p, \beta) \in \mathcal{S}$ ; hence  $\tilde{\mathcal{V}}_{(p_0, \beta_0)} = \tilde{\mathcal{V}} \subset \mathcal{S}$ . This proves that  $\mathcal{S}$  is open. Choosing any open set  $\mathcal{V} \subset \subset \tilde{\mathcal{V}}$  with  $(p_0, \beta_0) \in \mathcal{V}$ , the assertion (iii) will hold.

3. In this step, we continue Step 2 to deduce an equivalent condition for the invertibility of the Jacobian matrix  $D_{(\gamma', q', s')} F$  at  $(\gamma'_0, q'_0, s'_0; p_0, \beta_0)$ . By direct computation,

$$D_{(\gamma', q', s')} F = \begin{pmatrix} -s' I_n & M_{s'} & \omega_{s'}^- \\ -I_n & M_1 & 0 \\ q' & \gamma' & 0 \end{pmatrix} \in \mathbb{M}^{(n+n+1) \times (n+n+1)},$$

where  $I_n$  is the  $n \times n$  identity matrix,

$$M_{s'} = s' \left( \sigma'(|p + s' q'|) - \frac{\sigma(|p + s' q'|)}{|p + s' q'|} \right) \frac{p + s' q'}{|p + s' q'|} \otimes \frac{p + s' q'}{|p + s' q'|} + s' \frac{\sigma(|p + s' q'|)}{|p + s' q'|} I_n,$$

$$\omega_{s'}^{\pm} = \left( \sigma'(|p + s' q'|) - \frac{\sigma(|p + s' q'|)}{|p + s' q'|} \right) \left( \frac{p + s' q'}{|p + s' q'|} \cdot q' \right) \frac{p + s' q'}{|p + s' q'|} + \frac{\sigma(|p + s' q'|)}{|p + s' q'|} q' \pm \gamma'.$$

Here the prime only in  $\sigma'$  denotes the derivative. For notational simplicity, we write  $(\gamma', q', s'; p, \beta) = (\gamma'_0, q'_0, s'_0; p_0, \beta_0)$ . Applying suitable elementary row operations, as  $s' < 0$ , we have

$$D_{(\gamma', q', s')} F \rightarrow \begin{pmatrix} -s' I_n & M_{s'} & \omega_{s'}^- \\ O & M_1 - \frac{1}{s'} M_{s'} & -\frac{1}{s'} \omega_{s'}^- \\ 0 & \gamma' + \frac{q'_1}{s'} (M_{s'})^1 + \cdots + \frac{q'_n}{s'} (M_{s'})^n & \frac{1}{s'} q' \cdot \omega_{s'}^- \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -s' I_n & M_{s'} & \omega_{s'}^- \\ O & s' M_1 - M_{s'} & -\omega_{s'}^- \\ 0 & s' \gamma' + q'_1 (M_{s'})^1 + \cdots + q'_n (M_{s'})^n & q' \cdot \omega_{s'}^- \end{pmatrix},$$

where  $O$  is the  $n \times n$  zero matrix, and  $(M_{s'})^i$  is the  $i$ th row of  $M_{s'}$ . Since  $|q'| = -t_0$ ,  $\gamma' \cdot q' = 0$ , and  $s_+(r - \mu) < |p + s' q'| < s_+(r + \mu)$ , we have

$$q' \cdot \omega_{s'}^- = \left( \sigma'(|p + s' q'|) - \frac{\sigma(|p + s' q'|)}{|p + s' q'|} \right) \left( \frac{p + s' q'}{|p + s' q'|} \cdot q' \right)^2 + \frac{\sigma(|p + s' q'|)}{|p + s' q'|} t_0^2$$

$$= t_0^2 \left( \cos^2 \theta' \sigma'(|p + s' q'|) + (1 - \cos^2 \theta') \frac{\sigma(|p + s' q'|)}{|p + s' q'|} \right) > 0,$$

where  $\theta' \in [0, \pi]$  is the angle between  $p + s'q'$  and  $q'$ . After some elementary column operations to the last matrix from the above row operations, we obtain

$$D_{(\gamma', q', s')} F \rightarrow \begin{pmatrix} -s' I_n & M_{s'} - N_{s'} & \omega_{s'}^- \\ O & s' M_1 - M_{s'} + N_{s'} & -\omega_{s'}^- \\ 0 & 0 & q' \cdot \omega_{s'}^- \end{pmatrix},$$

where the  $j$ th column of  $N_{s'} \in \mathbb{M}^{n \times n}$  is  $\frac{s' \gamma'_j + q' \cdot (M_{s'})_j}{q' \cdot \omega_{s'}^-} \omega_{s'}^-$ . So  $D_{(\gamma', q', s')} F$  is invertible if and only if the  $n \times n$  matrix  $M_1 - \frac{1}{s'} M_{s'} + \frac{1}{s'} N_{s'}$  is invertible. We compute

$$\begin{aligned} M_1 - \frac{1}{s'} M_{s'} + \frac{1}{s'} N_{s'} &= \left( \sigma'(|p + q'|) - \frac{\sigma(|p + q'|)}{|p + q'|} \right) \frac{p + q'}{|p + q'|} \otimes \frac{p + q'}{|p + q'|} \\ &+ \frac{\sigma(|p + q'|)}{|p + q'|} I_n - \left( \sigma'(|p + s'q'|) - \frac{\sigma(|p + s'q'|)}{|p + s'q'|} \right) \frac{p + s'q'}{|p + s'q'|} \otimes \frac{p + s'q'}{|p + s'q'|} - \frac{\sigma(|p + s'q'|)}{|p + s'q'|} I_n \\ &+ \frac{1}{q' \cdot \omega_{s'}^-} \omega_{s'}^- \otimes \left[ \gamma' + \left( \sigma'(|p + s'q'|) - \frac{\sigma(|p + s'q'|)}{|p + s'q'|} \right) \left( \frac{p + s'q'}{|p + s'q'|} \cdot q' \right) \frac{p + s'q'}{|p + s'q'|} + \frac{\sigma(|p + s'q'|)}{|p + s'q'|} q' \right] \\ &= (a_1 - a_{s'}) I_n + (b_1 - a_1) \frac{p + q'}{|p + q'|} \otimes \frac{p + q'}{|p + q'|} \\ &\quad - (b_{s'} - a_{s'}) \frac{p + s'q'}{|p + s'q'|} \otimes \frac{p + s'q'}{|p + s'q'|} + \frac{1}{q' \cdot \omega_{s'}^-} \omega_{s'}^- \otimes \omega_{s'}^+, \end{aligned}$$

where

$$a_l = \frac{\sigma(|p + lq'|)}{|p + lq'|}, \quad b_l = \sigma'(|p + lq'|) \quad \text{for } l = s', 1.$$

Since  $a_1 < 0 < a_{s'}$ , we can set

$$\begin{aligned} B = B_{(p_0, \beta_0)} &= \frac{1}{a_1 - a_{s'}} (M_1 - \frac{1}{s'} M_{s'} + \frac{1}{s'} N_{s'}) \\ &= I_n + \frac{b_1 - a_1}{a_1 - a_{s'}} \frac{p + q'}{|p + q'|} \otimes \frac{p + q'}{|p + q'|} \\ &\quad - \frac{b_{s'} - a_{s'}}{a_1 - a_{s'}} \frac{p + s'q'}{|p + s'q'|} \otimes \frac{p + s'q'}{|p + s'q'|} + \frac{1}{(a_1 - a_{s'}) q' \cdot \omega_{s'}^-} \omega_{s'}^- \otimes \omega_{s'}^+; \end{aligned}$$

then  $D_{(\gamma', q', s')} F$  is invertible if and only if the matrix  $B \in \mathbb{M}^{n \times n}$  is invertible.

4. To close the arguments in Step 2 and thus to finish the proof, we choose a suitable  $\mu'_r > 0$  with  $0 < r - \mu'_r < r + \mu'_r < \sigma(s_+)$  in such a way that for each  $0 < \mu \leq \mu'_r$ , the matrix  $B = B_{(p_0, \beta_0)}$ , determined through Steps 2 and 3 for any given  $(p_0, \beta_0) \in \mathcal{S} = \mathcal{S}_{r-\mu, r+\mu}$ , is invertible.

First, let  $0 < \mu \leq \mu'_r \leq \mu_r$ , where the number  $\mu_r > 0$  is determined by Theorem 3.4. By Hypothesis (NF),

$$\frac{\sigma(l)}{l} < 0 < \frac{\sigma(k)}{k} \quad \forall k \in [s_+(r - \mu), s_+(r + \mu)],$$

$\forall l \in [s_-^2(r + \mu), s_-^2(r - \mu)]$  (**Type I**),  $\forall l \in [s_-^1(r - \mu), s_-^1(r + \mu)]$  (**Type II**).

So we can define a real-valued continuous function (to express the determinant of the matrix  $B = B_{(p_0, \beta_0)}$  from Step 3)

$$\begin{aligned} \text{DET}(v, u, q, \gamma) = & \det \left( I_n + \frac{\sigma'(|u|) - \frac{\sigma(|u|)}{|u|}}{\frac{\sigma(|u|)}{|u|} - \frac{\sigma(|v|)}{|v|}} \frac{u}{|u|} \otimes \frac{u}{|u|} - \frac{\sigma'(|v|) - \frac{\sigma(|v|)}{|v|}}{\frac{\sigma(|u|)}{|u|} - \frac{\sigma(|v|)}{|v|}} \frac{v}{|v|} \otimes \frac{v}{|v|} \right. \\ & + \frac{1}{(\frac{\sigma(|u|)}{|u|} - \frac{\sigma(|v|)}{|v|})(\sigma'(|v|) - \frac{\sigma(|v|)}{|v|})(\frac{v}{|v|} \cdot q)^2 + \frac{\sigma(|v|)}{|v|})} ((\sigma'(|v|) - \frac{\sigma(|v|)}{|v|})(\frac{v}{|v|} \cdot q) \frac{v}{|v|} \\ & \left. + \frac{\sigma(|v|)}{|v|} q - \gamma) \otimes ((\sigma'(|v|) - \frac{\sigma(|v|)}{|v|})(\frac{v}{|v|} \cdot q) \frac{v}{|v|} + \frac{\sigma(|v|)}{|v|} q + \gamma) \right) \end{aligned}$$

on the compact set  $\mathcal{M}$  of points  $(v, u, q, \gamma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n$  with

$$|v| \in [s_+(r - \mu), s_+(r + \mu)], \quad |\gamma| \leq 1,$$

$$|u| \in [s_-^2(r + \mu), s_-^2(r - \mu)] \quad (\textbf{Type I}), \quad |u| \in [s_-^1(r - \mu), s_-^1(r + \mu)] \quad (\textbf{Type II}).$$

With  $\bar{k} = s_+(r)$  and  $\bar{l} = -s_-^2(r)$  (**Type I**),  $\bar{l} = -s_-^1(r)$  (**Type II**), for each  $q \in \mathbb{S}^{n-1}$ ,

$$\text{DET}(\bar{k}q, \bar{l}q, q, 0) = \det \left( I_n + \frac{\sigma'(-\bar{l}) - \frac{\sigma(-\bar{l})}{-\bar{l}} + \frac{\sigma(\bar{k})}{\bar{k}}}{\frac{\sigma(-\bar{l})}{-\bar{l}} - \frac{\sigma(\bar{k})}{\bar{k}}} q \otimes q \right) \neq 0,$$

since  $\sigma'(-\bar{l}) \neq 0$  and hence the fraction in front of  $q \otimes q$  is different from  $-1$ . So

$$d := \min_{q \in \mathbb{S}^{n-1}} |\text{DET}(\bar{k}q, \bar{l}q, q, 0)| > 0.$$

Next, choose a number  $\rho > 0$  such that for all  $(v, u, q, \gamma), (\tilde{v}, \tilde{u}, \tilde{q}, \tilde{\gamma}) \in \mathcal{M}$  with  $|v - \tilde{v}|, |u - \tilde{u}|, |q - \tilde{q}|, |\gamma - \tilde{\gamma}| < \rho$ , we have

$$(3.6) \quad |\text{DET}(v, u, q, \gamma) - \text{DET}(\tilde{v}, \tilde{u}, \tilde{q}, \tilde{\gamma})| < d/2.$$

Let  $\mu'_r \in (0, \mu_r]$  be sufficiently small so that for all  $0 < \mu \leq \mu'_r$ ,

$$\begin{aligned} & h(s_-^2(r), s_+(r), r; s_-^2(r) - s_-^2(r + \mu), s_-^2(r - \mu) - s_-^2(r), \\ & \quad s_+(r) - s_+(r - \mu), s_+(r + \mu) - s_+(r), \mu, \mu) < \tau, \quad (\textbf{Type I}) \end{aligned}$$

$$\begin{aligned} & h(s_-^1(r), s_+(r), r; s_-^1(r) - s_-^1(r - \mu), s_-^1(r + \mu) - s_-^1(r), \\ & \quad s_+(r) - s_+(r - \mu), s_+(r + \mu) - s_+(r), \mu, \mu) < \tau, \quad (\textbf{Type II}) \end{aligned}$$

where  $h$  is the function in Theorem 3.4, and

$$\tau := \min\{\rho, \rho(s_+(r - \mu_r) - s_-^2(r - \mu_r))/4\}, \quad (\textbf{Type I})$$

$$\tau := \min\{\rho, \rho(s_+(r - \mu_r) - s_-^1(r + \mu_r))/4\}. \quad (\textbf{Type II})$$

Now, fix any  $\mu \in (0, \mu'_r]$ , and let  $B = B_{(p_0, \beta_0)}$  be the  $n \times n$  matrix determined through Steps 2 and 3 in terms of any given  $(p_0, \beta_0) \in \mathcal{S} = \mathcal{S}_{r-\mu, r+\mu}$ . Let  $p_+ = p_0 + s_0 q_0$  and  $p_- = p_0 + t_0 q_0$  from Step 2; then  $p_\pm$  and  $A(p_\pm)$  fulfill the conditions in Theorem 3.4. So this theorem implies that there exists a vector  $\zeta^0 \in \mathbb{S}^{n-1}$  such that

$$\max\{|p_-^0 - p_-|, |p_+^0 - p_+|, |A(p_-^0) - A(p_-)|, |A(p_+^0) - A(p_+)|\} < \tau,$$

where  $p_+^0 = \bar{k}\zeta^0$ ,  $p_-^0 = \bar{l}\zeta^0$ , and  $A(p_\pm^0) = r\zeta^0$ . Using (3.4) and (3.5),

$$\begin{aligned} |p_+ - \bar{k}\zeta^0| &< \rho, \quad |p_- - \bar{l}\zeta^0| < \rho, \\ |q_0 - \zeta^0| &= \left| \frac{p_+ - p_-}{s_0 - t_0} - \zeta^0 \right| \leq \frac{|(p_+ - p_-) - (\bar{k} - \bar{l})\zeta^0| + |(\bar{k} - \bar{l}) - (s_0 - t_0)|}{s_0 - t_0} \\ &\leq \frac{2\tau + ||p_+^0 - p_-^0| - |p_+ - p_-||}{s_0 - t_0} < \frac{4\tau}{s_0 - t_0} < \rho, \\ |\gamma_0| &= \left| \frac{A(p_+) - A(p_-)}{s_0 - t_0} \right| \leq \frac{|A(p_+) - A(p_+^0)| + |A(p_-^0) - A(p_-)|}{s_0 - t_0} < \rho. \end{aligned}$$

Since  $\det(B) = \text{DET}(p_+, p_-, q_0, \gamma_0)$  and  $|\text{DET}(\bar{k}\zeta^0, \bar{l}\zeta^0, \zeta^0, 0)| \geq d$ , it follows from (3.6) that

$$|\det(B)| > d/2 > 0.$$

The proof is now complete.  $\square$

**3.2. Relaxation of  $\nabla\omega(z) \in K_0$ .** The following result is important for the convex integration with a linear constraint; its proof can be found in [8, Lemma 4.5].

**Lemma 3.6.** *Let  $\lambda_1, \lambda_2 > 0$  and  $\eta_1 = -\lambda_1\eta$ ,  $\eta_2 = \lambda_2\eta$  with*

$$\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}, \quad |q| = 1, \quad \gamma \cdot q = 0, \quad b \neq 0.$$

*Let  $G \subset \mathbb{R}^{n+1}$  be a bounded domain. Then for each  $\epsilon > 0$ , there exists a function  $\omega = (\varphi, \psi) \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{1+n})$  with  $\text{supp}(\omega) \subset\subset G$  that satisfies the following properties:*

- (a)  $\text{div } \psi = 0$  in  $G$ ,
- (b)  $|\{z \in G \mid \nabla\omega(z) \notin \{\eta_1, \eta_2\}\}| < \epsilon$ ,
- (c)  $\text{dist}(\nabla\omega(z), [\eta_1, \eta_2]) < \epsilon$  for all  $z \in G$ ,
- (d)  $\|\omega\|_{L^\infty(G)} < \epsilon$ ,
- (e)  $\int_{\mathbb{R}^n} \varphi(x, t) dx = 0$  for all  $t \in \mathbb{R}$ .

We now state the relaxation theorem for homogeneous differential inclusion  $\nabla\omega(z) \in K_0$  in a form that is suitable for a later use; we restrict the inclusion only to the diagonal components.

**Theorem 3.7.** *Let  $0 < r < \sigma(s_+)$ , and let  $0 < \mu \leq \mu'_r$  for some number  $\mu'_r > 0$  with  $0 < r - \mu'_r < r + \mu'_r < \sigma(s_+)$  from Theorem 3.5. Let  $\mathcal{K}$  be a compact subset of  $\mathcal{S} = \mathcal{S}_{r-\mu, r+\mu}$ , and let  $\tilde{Q} \times \tilde{I}$  be a box in  $\mathbb{R}^{n+1}$ . Then given any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for each box  $Q \times I \subset \tilde{Q} \times \tilde{I}$ , point  $(p, \beta) \in \mathcal{K}$ , and number  $\rho > 0$  sufficiently small, there exists a function  $\omega = (\varphi, \psi) \in C_c^\infty(Q \times I; \mathbb{R}^{1+n})$  satisfying the following properties:*

- (a)  $\text{div } \psi = 0$  in  $Q \times I$ ,
- (b)  $(p' + D\varphi(z), \beta' + \psi_t(z)) \in \mathcal{S}$  for all  $z \in Q \times I$  and  $|(p', \beta') - (p, \beta)| \leq \delta$ ,
- (c)  $\|\omega\|_{L^\infty(Q \times I)} < \rho$ ,
- (d)  $\int_{Q \times I} |\beta + \psi_t(z) - A(p + D\varphi(z))| dz < \epsilon |Q \times I| / |\tilde{Q} \times \tilde{I}|$ ,
- (e)  $\int_{Q \times I} \text{dist}((p + D\varphi(z), \beta + \psi_t(z)), \mathcal{A}) dz < \epsilon |Q \times I| / |\tilde{Q} \times \tilde{I}|$ ,

$$(f) \quad \int_Q \varphi(x, t) dx = 0 \text{ for all } t \in I,$$

$$(g) \quad \|\varphi_t\|_{L^\infty(Q \times I)} < \rho,$$

where  $\mathcal{A} = \mathcal{A}_{r-\mu, r+\mu} \subset \mathbb{R}^{n+n}$  is the set defined by

$$\mathcal{A} = \{(p', A(p')) \mid |p'| \in [s_-^2(r+\mu), s_-^2(r-\mu)] \cup [s_+(r-\mu), s_+(r+\mu)]\}, \text{ (Type I)}$$

$$\mathcal{A} = \{(p', A(p')) \mid |p'| \in [s_-^1(r-\mu), s_-^1(r+\mu)] \cup [s_+(r-\mu), s_+(r+\mu)]\}. \text{ (Type II)}$$

*Proof.* By Theorem 3.5, there exist finitely many open balls  $\mathcal{B}_1, \dots, \mathcal{B}_N \subset \mathcal{S}$  covering  $\mathcal{K}$  and  $C^1$  functions  $q_i : \bar{\mathcal{B}}_i \rightarrow \mathbb{S}^{n-1}$ ,  $\gamma_i : \bar{\mathcal{B}}_i \rightarrow \mathbb{R}^n$ ,  $t_{i,\pm} : \bar{\mathcal{B}}_i \rightarrow \mathbb{R}$  ( $1 \leq i \leq N$ ) with  $\gamma_i \cdot q_i = 0$  and  $t_{i,-} < 0 < t_{i,+}$  on  $\bar{\mathcal{B}}_i$  such that for each  $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0) = R(F_{0,r-\mu,r+\mu})$  with  $(p, \beta) \in \bar{\mathcal{B}}_i$ , we have

$$\xi + t_{i,\pm} \eta_i \in F_\pm = F_{\pm, r-\mu, r+\mu},$$

where  $t_{i,\pm} = t_{i,\pm}(p, \beta)$ ,  $\eta_i = \begin{pmatrix} q_i(p, \beta) & b \\ \frac{1}{b} \gamma_i(p, \beta) \otimes q_i(p, \beta) & \gamma_i(p, \beta) \end{pmatrix}$ , and  $b \neq 0$  is arbitrary.

Let  $1 \leq i \leq N$ . We write  $\xi_i = \xi_i(p, \beta) = \begin{pmatrix} p & 0 \\ O & \beta \end{pmatrix} \in R(F_0)$  for  $(p, \beta) \in \bar{\mathcal{B}}_i \subset \mathcal{S}$ , where  $O$  is the  $n \times n$  zero matrix. We omit the dependence on  $(p, \beta) \in \bar{\mathcal{B}}_i$  in the following whenever it is clear from the context. Given any  $\rho > 0$ , we choose a constant  $b_i = b_{i,\rho}$  with

$$0 < b_i < \min_{\bar{\mathcal{B}}_i} \frac{\rho}{t_{i,+} - t_{i,-}}.$$

With this choice of  $b = b_i$ , let  $\eta_i$  be defined on  $\bar{\mathcal{B}}_i$  as above. Then

$$\xi_{i,\pm} = \begin{pmatrix} p_{i,\pm} & c_{i,\pm} \\ B_{i,\pm} & \beta_{i,\pm} \end{pmatrix} := \xi_i + t_{i,\pm} \eta_i \in F_\pm,$$

$$\xi_i = \lambda_i \xi_{i,+} + (1 - \lambda_i) \xi_{i,-}, \quad \lambda_i = \frac{-t_{i,-}}{t_{i,+} - t_{i,-}} \in (0, 1) \quad \text{on } \bar{\mathcal{B}}_i.$$

By the definition of  $R(F_0)$ , on  $\bar{\mathcal{B}}_i$ , both  $\xi_{i,-}^\tau = \tau \xi_{i,+} + (1 - \tau) \xi_{i,-}$  and  $\xi_{i,+}^\tau = (1 - \tau) \xi_{i,+} + \tau \xi_{i,-}$  belong to  $R(F_0)$  for all  $\tau \in (0, 1)$ . Let  $0 < \tau < \min_{1 \leq j \leq N} \min_{\bar{\mathcal{B}}_j} \min\{\lambda_j, 1 - \lambda_j\} \leq \frac{1}{2}$  be a small number to be selected later. Let  $\lambda'_i = \frac{\lambda_i - \tau}{1 - 2\tau}$  on  $\bar{\mathcal{B}}_i$ . Then  $\lambda'_i \in (0, 1)$  and  $\xi_i = \lambda'_i \xi_{i,+}^\tau + (1 - \lambda'_i) \xi_{i,-}^\tau$  on  $\bar{\mathcal{B}}_i$ . Moreover, on  $\bar{\mathcal{B}}_i$ ,  $\xi_{i,+}^\tau - \xi_{i,-}^\tau = (1 - 2\tau)(\xi_{i,+} - \xi_{i,-})$  is rank-one,  $[\xi_{i,-}^\tau, \xi_{i,+}^\tau] \subset (\xi_{i,-}, \xi_{i,+}) \subset R(F_0)$ , and

$$c\tau \leq |\xi_{i,+}^\tau - \xi_{i,-}^\tau| = |\xi_{i,-}^\tau - \xi_{i,-}| = \tau |\xi_{i,+} - \xi_{i,-}| = \tau(t_{i,+} - t_{i,-}) |\eta_i| \leq C\tau,$$

where  $C = \max_{1 \leq j \leq N} \max_{\bar{\mathcal{B}}_j} (t_{j,+} - t_{j,-}) |\eta_j| \geq \min_{1 \leq j \leq N} \min_{\bar{\mathcal{B}}_j} (t_{j,+} - t_{j,-}) |\eta_j| = c > 0$ . By continuity,  $H_\tau = \bigcup_{(p,\beta) \in \bar{\mathcal{B}}_i, 1 \leq j \leq N} [\xi_{j,-}^\tau(p, \beta), \xi_{j,+}^\tau(p, \beta)]$  is a compact subset of  $R(F_0)$ , where  $R(F_0)$  is open in the space

$$\Sigma_0 = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \mid \text{tr} B = 0 \right\},$$

by Lemma 3.2 and Theorem 3.5. So  $d_\tau = \text{dist}(H_\tau, \partial|_{\Sigma_0} R(F_0)) > 0$ , where  $\partial|_{\Sigma_0}$  is the relative boundary in  $\Sigma_0$ .

Let  $\eta_{i,1} = -\lambda_{i,1}\eta_i = -\lambda'_i(1-2\tau)(t_{i,+} - t_{i,-})\eta_i$ ,  $\eta_{i,2} = \lambda_{i,2}\eta_i = (1-\lambda'_i)(1-2\tau)(t_{i,+} - t_{i,-})\eta_i$  on  $\bar{B}_i$ , where  $\lambda_{i,1} = \tau(-t_{i,+}) + (1-\tau)(-t_{i,-}) > 0$ ,  $\lambda_{i,2} = (1-\tau)t_{i,+} + \tau t_{i,-} > 0$  on  $\bar{B}_i$ , and  $\tau > 0$  is so small that

$$\min_{1 \leq j \leq N} \min_{\bar{B}_j} \lambda_{j,k} > 0 \quad (k = 1, 2).$$

Applying Lemma 3.6 to matrices  $\eta_{i,1} = \eta_{i,1}(p, \beta)$ ,  $\eta_{i,2} = \eta_{i,2}(p, \beta)$  (also depending on  $\rho$ ) for a fixed  $(p, \beta) \in \bar{B}_i$  and to a given box  $G = Q \times I$ , we obtain that for each  $\rho > 0$ , there exist a function  $\omega = (\varphi, \psi) \in C_c^\infty(Q \times I; \mathbb{R}^{1+n})$  and an open set  $G_\rho \subset\subset Q \times I$  satisfying the following conditions:

(3.7)

$$\begin{cases} (1) & \text{div } \psi = 0 \text{ in } Q \times I, \\ (2) & |(Q \times I) \setminus G_\rho| < \rho; \quad \xi_i + \nabla \omega(z) \in \{\xi_{i,-}^\tau, \xi_{i,+}^\tau\} \text{ for all } z \in G_\rho, \\ (3) & \xi_i + \nabla \omega(z) \in [\xi_{i,-}^\tau, \xi_{i,+}^\tau]_\rho \text{ for all } z \in Q \times I, \\ (4) & \|\omega\|_{L^\infty(Q \times I)} < \rho, \\ (5) & \int_Q \varphi(x, t) dx = 0 \text{ for all } t \in I, \\ (6) & \|\varphi_t\|_{L^\infty(Q \times I)} < 2\rho, \end{cases}$$

where  $[\xi_{i,-}^\tau, \xi_{i,+}^\tau]_\rho$  denotes the  $\rho$ -neighborhood of the closed line segment  $[\xi_{i,-}^\tau, \xi_{i,+}^\tau]$ . Here, from (3.7.3), condition (3.7.6) follows as

$$|\varphi_t| < |c_{i,+} - c_{i,-}| + \rho = (t_{i,+} - t_{i,-})|b_i| + \rho < 2\rho \quad \text{in } Q \times I.$$

Note (a), (c), (f), and (g) follow from (3.7), where  $2\rho$  in (3.7.6) can be adjusted to  $\rho$  as in (g). By the uniform continuity of  $A$  on the set  $J = \{p' \in \mathbb{R}^n \mid |p'| \leq s_+\}$ , we can find a  $\delta' > 0$  such that  $|A(p') - A(p'')| < \frac{\epsilon}{3|Q \times I|}$  whenever  $p', p'' \in J$  and  $|p' - p''| < \delta'$ . We then choose a  $\tau > 0$  so small that

$$C\tau < \delta', \quad C|\tilde{Q} \times \tilde{I}|\tau < \frac{\epsilon}{3}.$$

Next, we choose a  $\delta > 0$  such that  $\delta < \frac{d_\tau}{2}$ . If  $0 < \rho < \delta$ , then by (3.7.1) and (3.7.3), for all  $z \in Q \times I$  and  $|(p', \beta') - (p, \beta)| \leq \delta$ ,

$$\xi_i(p', \beta') + \nabla \omega(z) \in \Sigma_0, \quad \text{dist}(\xi_i(p', \beta') + \nabla \omega(z), H_\tau) < d_\tau,$$

and so  $\xi_i(p', \beta') + \nabla \omega(z) \in R(F_0)$ , that is,  $(p' + D\varphi(z), \beta' + \psi_t(z)) \in \mathcal{S}$ . Thus (b) holds for all  $0 < \rho < \delta$ . In particular,  $(p + D\varphi(z), \beta + \psi_t(z)) \in \mathcal{S}$  and so  $|p + D\varphi(z)| \leq s_+(r + \mu)$  and  $|\beta + \psi_t(z)| \leq r + \mu$  for all  $z \in Q \times I$ , by (i) of

Theorem 3.5. Thus

$$\begin{aligned}
& \int_{Q \times I} |\beta + \psi_t - A(p + D\varphi)| dz \\
& \leq \int_{G_\rho} |\beta + \psi_t - A(p + D\varphi)| dz + 2\sigma(s_+)\rho \\
& \leq |Q \times I| \max\{|\beta_{i,\pm}^\tau - A(p_{i,\pm}^\tau)|\} + 2\sigma(s_+)\rho \\
& \leq C|Q \times I|\tau + |Q \times I| \max\{|A(p_{i,\pm}) - A(p_{i,\pm}^\tau)|\} + 2\sigma(s_+)\rho \\
& \leq \frac{2\epsilon|Q \times I|}{3|\tilde{Q} \times \tilde{I}|} + 2\sigma(s_+)\rho,
\end{aligned}$$

where  $\xi_{i,\pm}^\tau = \begin{pmatrix} p_{i,\pm}^\tau & c_{i,\pm}^\tau \\ B_{i,\pm}^\tau & \beta_{i,\pm}^\tau \end{pmatrix}$ . Thus, (d) holds for all  $\rho > 0$  satisfying  $2\sigma(s_+)\rho < \frac{\epsilon|Q \times I|}{3|\tilde{Q} \times \tilde{I}|}$ . Similarly,

$$\begin{aligned}
& \int_{Q \times I} \text{dist}((p + D\varphi(z), \beta + \psi_t(z)), \mathcal{A}) dz \\
& \leq \int_{G_\rho} \max |(p_{i,\pm}^\tau, \beta_{i,\pm}^\tau) - (p_{i,\pm}, \beta_{i,\pm})| dz + 2(s_+ + \sigma(s_+))\rho \\
& \leq C|Q \times I|\tau + 2(s_+ + \sigma(s_+))\rho \\
& \leq \frac{\epsilon|Q \times I|}{3|\tilde{Q} \times \tilde{I}|} + 2(s_+ + \sigma(s_+))\rho;
\end{aligned}$$

therefore, (e) holds for all  $\rho > 0$  with  $(s_+ + \sigma(s_+))\rho < \frac{\epsilon|Q \times I|}{3|\tilde{Q} \times \tilde{I}|}$ .

We have verified (a) – (g) for any  $(p, \beta) \in \bar{\mathcal{B}}_i$  and  $1 \leq i \leq N$ , where  $\delta > 0$  is independent of the index  $i$ . Since  $\mathcal{B}_1, \dots, \mathcal{B}_N$  cover  $\mathcal{K}$ , the proof is now complete.  $\square$

#### 4. BOUNDARY FUNCTION $\Phi$ AND THE ADMISSIBLE SET $\mathcal{U}$ BY A COUNTABLE OPEN COVERING

To start the proof of Theorem 1.1, assume  $\Omega$  and  $u_0$  satisfy (1.5) and (1.6).

**4.1. Boundary function  $\Phi$ .** We first construct a suitable boundary function  $\Phi = (u^*, v^*)$  to prove Theorem 1.1 in the setting of the general existence theorem, Theorem 2.1. Assuming all the hypotheses in Theorem 1.1, we fix any  $\tilde{r} \in (\sigma(m'_0), \sigma(s_+))$ . For each  $r \in (0, \tilde{r})$ , let  $\bar{\mu}_r > 0$  be chosen so that

$$0 < r - \bar{\mu}_r < r + \bar{\mu}_r < \tilde{r} \quad \text{and} \quad \bar{\mu}_r \leq \mu'_r,$$

where  $\mu'_r > 0$  is some number from Theorem 3.7. Then  $\{I_r := (r - \bar{\mu}_r, r + \bar{\mu}_r)\}_{r \in (0, \tilde{r})}$  is an open covering for the interval  $(0, \tilde{r})$ . For convenience, we select a countable sub-covering  $\{I_{r_k}\}_{k \in \mathbb{N}}$  of  $\{I_r\}_{r \in (0, \tilde{r})}$  for  $(0, \tilde{r})$ . We now

define a *diagonal-covering set*  $\mathcal{S}_{dc} \subset \mathbb{R}^{n+n}$  by

$$\mathcal{S}_{dc} = \bigcup_{k \in \mathbb{N}} \mathcal{S}_{r_k - \bar{\mu}_{r_k}, r_k + \bar{\mu}_{r_k}};$$

then by Theorem 3.5,  $\mathcal{S}_{dc} \subset \mathbb{R}^{n+n}$  is open and bounded.

Next, we apply Lemma 2.3 to the number  $r = \tilde{r}$  in order to determine functions  $\tilde{\sigma}, \tilde{f} \in C^{1+\alpha}([0, \infty))$  satisfying its conclusion. Also, let  $\tilde{A}(p) = \tilde{f}(|p|^2)p$  ( $p \in \mathbb{R}^n$ ). Then the following holds.

**Lemma 4.1.** *We have*

$$(p, \tilde{A}(p)) \in \mathcal{S}_{dc} \quad \forall 0 < |p| < s_+(\tilde{r}).$$

*Proof.* Let  $0 < s = |p| < s_+(\tilde{r})$ ,  $r = \tilde{\sigma}(s)$  and  $\zeta = p/|p|$ , so that  $\zeta \in \mathbb{S}^{n-1}$ ,  $\tilde{A}(p) = r\zeta$ , and  $0 < r < \tilde{r}$ . Set  $p_+ = s_+(r)\zeta$ ,  $p_- = -s_-^2(r)\zeta$  (**Type I**),  $p_- = -s_-^1(r)\zeta$  (**Type II**), and  $\beta_{\pm} = r\zeta$ . Then  $A(p_{\pm}) = r\zeta = \beta_{\pm}$ . Define  $\xi = \begin{pmatrix} p & 0 \\ O & \tilde{A}(p) \end{pmatrix}$  and  $\xi_{\pm} = \begin{pmatrix} p_{\pm} & 0 \\ O & \beta_{\pm} \end{pmatrix}$ . Then  $\xi = \lambda\xi_+ + (1-\lambda)\xi_-$  for some  $0 < \lambda < 1$ .

Observe now that  $r \in I_{r_k}$  for some  $k \in \mathbb{N}$ , that is,  $r_k - \bar{\mu}_{r_k} < r < r_k + \bar{\mu}_{r_k}$ . We thus have

$$\xi_{\pm} \in F_{\pm, r_k - \bar{\mu}_{r_k}, r_k + \bar{\mu}_{r_k}}.$$

Since  $\text{rank}(\xi_+ - \xi_-) = 1$ , it follows from the definition of  $R(F_{0, r_k - \bar{\mu}_{r_k}, r_k + \bar{\mu}_{r_k}})$  that  $\xi \in (\xi_-, \xi_+) \subset R(F_{0, r_k - \bar{\mu}_{r_k}, r_k + \bar{\mu}_{r_k}})$ . Thus by Lemma 3.2,

$$(p, \tilde{A}(p)) \in \mathcal{S}_{r_k - \bar{\mu}_{r_k}, r_k + \bar{\mu}_{r_k}} \subset \mathcal{S}_{dc}.$$

□

By Lemma 2.3, equation  $u_t = \text{div}(\tilde{A}(Du))$  is uniformly parabolic. So by Theorem 2.2, the initial-Neumann boundary value problem

$$(4.1) \quad \begin{cases} u_t^* = \text{div}(\tilde{A}(Du^*)) & \text{in } \Omega_T \\ \partial u^*/\partial \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T) \\ u^*(x, 0) = u_0(x), & x \in \Omega \end{cases}$$

admits a unique classical solution  $u^* \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_T)$ .

From conditions (1.5) and (1.6), we can find a function  $h \in C^{2+\alpha}(\bar{\Omega})$  satisfying

$$\Delta h = u_0 \text{ in } \Omega \quad \text{and} \quad \partial h / \partial \mathbf{n} = 0 \text{ on } \partial\Omega.$$

Let  $v_0 = Dh \in C^{1+\alpha}(\bar{\Omega}; \mathbb{R}^n)$  and define, for  $(x, t) \in \Omega_T$ ,

$$(4.2) \quad v^*(x, t) = v_0(x) + \int_0^t \tilde{A}(Du^*(x, s)) ds.$$



Then it is easily seen that  $\Phi := (u^*, v^*) \in C^1(\bar{\Omega}_T; \mathbb{R}^{1+n})$  satisfies (2.1); that is,

$$(4.3) \quad \begin{cases} u^*(x, 0) = u_0(x) \ (x \in \Omega), \\ \operatorname{div} v^* = u^* \text{ in } \Omega_T, \\ v^*(\cdot, t) \cdot \mathbf{n}|_{\partial\Omega} = 0 \ \forall t \in [0, T], \end{cases}$$

and so  $\Phi$  is a boundary function for the initial datum  $u_0$ .

Next, let

$$\mathcal{F} = \{(p, A(p)) \mid |p| \in \{0\} \cup [s_+(\tilde{r}), \max\{M, s_+\}]\},$$

where  $M = \|Du^*\|_{L^\infty(\Omega_T)}$ . Then we have the following:

**Lemma 4.2.**

$$(Du^*(x, t), v_t^*(x, t)) \in \mathcal{S}_{dc} \cup \mathcal{F} \quad \forall (x, t) \in \Omega_T.$$

*Proof.* Let  $(x, t) \in \Omega_T$  and  $p = Du^*(x, t)$ .

First, assume  $|p| \in \{0\} \cup [s_+(\tilde{r}), \max\{M, s_+\}]$ . Then  $\tilde{A}(p) = A(p)$  and hence by (4.2)

$$(Du^*(x, t), v_t^*(x, t)) = (p, \tilde{A}(p)) = (p, A(p)) \in \mathcal{F}.$$

Otherwise, we have  $0 < |p| < s_+(\tilde{r})$ , and so by Lemma 4.1 and (4.2),

$$(Du^*(x, t), v_t^*(x, t)) = (p, \tilde{A}(p)) \in \mathcal{S}_{dc}.$$

□

We adopt the following terminology that is needed below.

**Definition 4.3.** Let  $G$  be an open set in  $\mathbb{R}^N$ . We say a function  $u$  is *piecewise  $C^1$*  in  $G$  and write  $u \in C_{piece}^1(G)$  if there exists a sequence of disjoint open sets  $\{G_j\}_{j=1}^\infty$  in  $G$  such that

$$u \in C^1(\bar{G}_j) \ \forall j \in \mathbb{N}, \quad |G \setminus \cup_{j=1}^\infty G_j| = 0.$$

It is then necessary to have  $|\partial G_j| = 0 \ \forall j \in \mathbb{N}$ .

**4.2. Selection of interface.** To separate the space-time domain  $\Omega_T$  into the classical and micro-oscillatory parts for Lipschitz solutions, we set

$$\begin{aligned} \Omega_T^0 &= \{(x, t) \in \Omega_T \mid |Du^*(x, t)| = 0\}, \\ \Omega_T^1 &= \{(x, t) \in \Omega_T \mid 0 < |Du^*(x, t)| < s_+(\tilde{r})\}, \\ \Omega_T^2 &= \{(x, t) \in \Omega_T \mid |Du^*(x, t)| = s_+(\tilde{r})\}, \\ \Omega_T^{\tilde{r}} &= \Omega_T^3 = \{(x, t) \in \Omega_T \mid |Du^*(x, t)| > s_+(\tilde{r})\}, \\ \Omega_0^{\tilde{r}} &= \{(x, 0) \mid x \in \Omega, |Du_0(x)| > s_+(\tilde{r})\}, \end{aligned}$$

then  $\Omega_T = \cup_{k=0}^3 \Omega_T^k$ ,  $\Omega_T^1 \neq \emptyset$ , and  $\Omega_0^{\tilde{r}} \subset \partial\Omega_T^{\tilde{r}}$ . Observe from the proof of Lemma 4.2 that  $(Du^*, v_t^*) \in \mathcal{S}_{dc}$  in  $\Omega_T^1$ .

**4.3. The admissible set  $\mathcal{U}$ .** Let  $m = \|u_t^*\|_{L^\infty(\Omega_T)} + 1$ . We define  $\mathcal{U}$  to be the set of all  $u \in W_{u^*}^{1,\infty}(\Omega_T)$  satisfying

$$(4.4) \quad \begin{cases} \|u_t\|_{L^\infty(\Omega_T)} < m, \\ \exists \text{ an open set } G \subset \subset \Omega_T^1 \text{ with } |\partial G| = 0 \text{ such that} \\ \quad u \equiv u^* \text{ in } \Omega_T \setminus G \text{ and } u \in C_{piece}^1(G), \text{ and} \\ \exists v \in W_{v^*}^{1,\infty}(\Omega_T; \mathbb{R}^n) \text{ such that } v \equiv v^* \text{ in } \Omega_T \setminus G, \\ \quad v \in C_{piece}^1(G; \mathbb{R}^n), \operatorname{div} v = u \text{ and } (Du, v_t) \in \mathcal{S}_{dc} \cup \mathcal{F} \text{ a.e. in } \Omega_T, \\ \text{and } (Du, v_t) \in \mathcal{S}_{dc} \text{ a.e. in } \Omega_T^1. \end{cases}$$

Next, for each  $\epsilon > 0$ , let  $\mathcal{U}_\epsilon$  be the set of all  $u \in \mathcal{U}$  satisfying, in addition to (4.4),

$$\begin{cases} \int_{\Omega_T} |v_t - A(Du)| dx dt \leq \epsilon |\Omega_T|, \\ \int_{\Omega_T^1} \operatorname{dist}((Du, v_t), \mathcal{B}) dx dt \leq \epsilon |\Omega_T^1|, \end{cases}$$

where  $\mathcal{B} = \mathcal{B}_{\tilde{r}} \subset \mathbb{R}^{n+n}$  is the set given by

$$\mathcal{B} = \{(p, A(p)) \mid |p| \in [s_-^2(\tilde{r}), s_+(\tilde{r})]\}, \quad (\text{Type I})$$

$$\mathcal{B} = \{(p, A(p)) \mid |p| \in [0, s_-^1(\tilde{r})] \cup [s_0, s_+(\tilde{r})]\}. \quad (\text{Type II})$$

Observe here that

$$\bigcup_{k \in \mathbb{N}} \mathcal{A}_{r_k - \bar{\mu}_{r_k}, r_k + \bar{\mu}_{r_k}} \subset \mathcal{B},$$

where the sets  $\mathcal{A}_{r_k - \bar{\mu}_{r_k}, r_k + \bar{\mu}_{r_k}}$  are as in Theorem 3.7.

**Remark 4.4.** Summarizing the above, we have constructed a boundary function  $\Phi = (u^*, v^*) \in C^1(\bar{\Omega}_T; \mathbb{R}^{1+n})$  for the initial datum  $u_0$  in such a way that the admissible set  $\mathcal{U}$  contains  $u^*$ ; so  $\mathcal{U}$  is nonempty. Also  $\mathcal{U}$  is a bounded subset of  $W_{u^*}^{1,\infty}(\Omega_T)$ , since  $\mathcal{S}_{dc} \cup \mathcal{F}$  is bounded and  $\|u_t\|_{L^\infty(\Omega_T)} < m$  for all  $u \in \mathcal{U}$ . Moreover, by (i) of Theorem 3.5 and the definition of  $\mathcal{F}$ , for each  $u \in \mathcal{U}$ , its corresponding vector function  $v$  satisfies  $\|v_t\|_{L^\infty(\Omega_T)} \leq \max\{\sigma(M_0), \sigma(s_+)\}$ ; this bound plays the role of a fixed number  $R > 0$  in the general density approach in Subsection 2.1. Finally, note that  $\tilde{A}(Du^*) \neq A(Du^*)$  in the nonempty open set  $\Omega_T^1$ ; hence  $u^*$  itself is not a Lipschitz solution to (1.1).

In view of the general existence theorem, Theorem 2.1, it only remains to prove the  $L^\infty$ -density of  $\mathcal{U}_\epsilon$  in  $\mathcal{U}$  towards the existence of infinitely many Lipschitz solutions to problem (1.1) for both types; this core subject is carried out in the next section.

## 5. DENSITY OF $\mathcal{U}_\epsilon$ IN $\mathcal{U}$ :

### FINAL STEP FOR THE PROOF OF THEOREM 1.1

In this section, we follow Section 4 to complete the proof of Theorem 1.1.

**5.1. The density property.** The density theorem below is the last preparation for both **Types I** and **II**.

**Theorem 5.1.** *For each  $\epsilon > 0$ ,  $\mathcal{U}_\epsilon$  is dense in  $\mathcal{U}$  under the  $L^\infty$ -norm.*

*Proof.* Given any  $\epsilon > 0$ , let  $u \in \mathcal{U}$  and  $\eta > 0$ . The goal is to construct a function  $\tilde{u} \in \mathcal{U}_\epsilon$  such that  $\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta$ . For clarity, we divide the proof into several steps.

1. Note from (4.4) that  $\|u_t\|_{L^\infty(\Omega_T)} < m - \bar{\tau}_0$  for some  $\bar{\tau}_0 > 0$ , there exists an open set  $G \subset \subset \Omega_T^1$  with  $|\partial G| = 0$  such that  $u \equiv u^*$  in  $\Omega_T \setminus G$  and  $u \in C_{piece}^1(G)$ , and there exists a vector function  $v \in W_{v^*}^{1,\infty}(\Omega_T; \mathbb{R}^n)$  such that  $v \equiv v^*$  in  $\Omega_T \setminus G$ ,  $v \in C_{piece}^1(G; \mathbb{R}^n)$ ,  $\operatorname{div} v = u$  and  $(Du, v_t) \in \mathcal{S}_{dc} \cup \mathcal{F}$  a.e. in  $\Omega_T$ , and  $(Du, v_t) \in \mathcal{S}_{dc}$  a.e. in  $\Omega_T^1$ . Since both  $u$  and  $v$  are piecewise  $C^1$  in  $G$ , there exists a sequence of disjoint open sets  $\{G_j\}_{j=1}^\infty$  in  $G$  with  $|\partial G_j| = 0$  such that

$$u \in C^1(\bar{G}_j), v \in C^1(\bar{G}_j; \mathbb{R}^n) \quad \forall j \geq 1, \quad |G \setminus \cup_{j=1}^\infty G_j| = 0.$$

We also choose an open set  $G_0 \subset \subset \Omega_T^1 \setminus \bar{G}$  with  $|\partial G_0| = 0$  such that

$$(5.1) \quad \begin{cases} \int_{(\Omega_T^1 \setminus \bar{G}) \setminus G_0} |v_t - A(Du)| dz \leq \frac{\epsilon}{5} |\Omega_T^1|, \\ \int_{(\Omega_T^1 \setminus \bar{G}) \setminus G_0} \operatorname{dist}((Du, v_t), \mathcal{B}) dz \leq \frac{\epsilon}{5} |\Omega_T^1|. \end{cases}$$

Then the open set  $\tilde{G} := G_0 \cup G$  is such that  $\tilde{G} \subset \subset \Omega_T^1$ ,  $|\partial \tilde{G}| = 0$ , and  $\{G_j\}_{j=0}^\infty$  is a sequence of disjoint open subsets of  $\tilde{G}$  whose union has measure  $|\tilde{G}|$ .

2. Let  $j \in \{0\} \cup \mathbb{N} =: \mathbb{N}_0$  be fixed. Note that  $(Du(z), v_t(z)) \in \bar{\mathcal{S}}_{dc}$  for all  $z = (x, t) \in G_j$  and that  $H_j := \{z \in G_j \mid (Du(z), v_t(z)) \in \partial \mathcal{S}_{dc}\}$  is a (relatively) closed set in  $G_j$  with measure zero. So  $G'_j := G_j \setminus H_j$  is an open subset of  $G_j$  with  $|G'_j| = |G_j|$ , and  $(Du(z), v_t(z)) \in \mathcal{S}_{dc}$  for all  $z \in G'_j$ . Now, we choose an open set  $G''_j \subset \subset G'_j$  with  $|\partial G''_j| = 0$  such that

$$(5.2) \quad \begin{cases} \int_{G'_j \setminus G''_j} |v_t - A(Du)| dz \leq \frac{\epsilon}{5 \cdot 2^{j+1}} |\Omega_T^1|, \\ \int_{G'_j \setminus G''_j} \operatorname{dist}((Du, v_t), \mathcal{B}) dz \leq \frac{\epsilon}{5 \cdot 2^{j+1}} |\Omega_T^1|. \end{cases}$$

Observe that  $(Du(z), v_t(z)) \in \mathcal{S}_{dc}$  for all  $z$  in the compact set  $\bar{G}''_j$ ; so we are able to choose a finite index set  $K_j \subset \mathbb{N}$  such that

$$(Du(z), v_t(z)) \in \bigcup_{k \in K_j} \mathcal{S}_{r_k - \bar{\mu}_{r_k}, r_k + \bar{\mu}_{r_k}} \quad \forall z \in \bar{G}''_j.$$

Let  $k_1^j < \dots < k_{n_j}^j$  denote the indices in  $K_j$ , where  $n_j$  is the cardinality of  $K_j$ , and write

$$\mathcal{S}_l^j = \mathcal{S}_{r_{k_l^j} - \bar{\mu}_{r_{k_l^j}}, r_{k_l^j} + \bar{\mu}_{r_{k_l^j}}} \quad \text{for } l = 1, \dots, n_j.$$

3. For each  $l \in \{1, \dots, n_j\}$  and each  $\tau > 0$ , let

$$\mathcal{G}_{l,\tau}^j = \left\{ (p, \beta) \in \mathcal{S}_l^j \mid \text{dist}((p, \beta), \partial \mathcal{S}_l^j) > \tau, \text{dist}((p, \beta), \mathcal{B}) > \tau \right\};$$

then

$$\mathcal{S}_l^j = (\cup_{\tau>0} \mathcal{G}_{l,\tau}^j) \cup \{(p, \beta) \in \mathcal{S}_l^j \mid \text{dist}((p, \beta), \mathcal{B}) = 0\}.$$

Note

$$\begin{aligned} & \int_{G_j''} |v_t(z) - A(Du(z))| dz \\ & \leq \sum_{l=1}^{n_j} \int_{\{z \in G_j'' \mid (Du(z), v_t(z)) \in \mathcal{S}_l^j\}} |v_t(z) - A(Du(z))| dz \\ & = \sum_{l=1}^{n_j} \lim_{\tau \rightarrow 0^+} \int_{\{z \in G_j'' \mid (Du(z), v_t(z)) \in \mathcal{G}_{l,\tau}^j\}} |v_t(z) - A(Du(z))| dz, \\ & \int_{G_j''} \text{dist}((Du(z), v_t(z)), \mathcal{B}) dz \\ & \leq \sum_{l=1}^{n_j} \lim_{\tau \rightarrow 0^+} \int_{\{z \in G_j'' \mid (Du(z), v_t(z)) \in \mathcal{G}_{l,\tau}^j\}} \text{dist}((Du(z), v_t(z)), \mathcal{B}) dz; \end{aligned}$$

then we choose a  $\tau_j > 0$  so that for  $l = 1, \dots, n_j$ , we have  $|\partial O_l^j| = 0$  and

$$(5.3) \quad \begin{cases} \int_{F_l^j} |v_t(z) - A(Du(z))| dz < \frac{\epsilon}{5 \cdot 2^{j+1} n_j} |\Omega_T^1|, \\ \int_{F_l^j} \text{dist}((Du(z), v_t(z)), \mathcal{B}) dz < \frac{\epsilon}{5 \cdot 2^{j+1} n_j} |\Omega_T^1|, \end{cases}$$

where  $O_l^j = \{z \in G_j'' \mid (Du(z), v_t(z)) \in \mathcal{G}_{l,\tau_j}^j\}$  and  $F_l^j = G_j'' \setminus O_l^j$ . We also define  $U_1^j = O_1^j$  and  $U_l^j = O_l^j \setminus (\bar{O}_1^j \cup \dots \cup \bar{O}_{l-1}^j)$  ( $l = 2, \dots, n_j$ ); then  $U_1^j, \dots, U_{n_j}^j$  are disjoint open subsets of  $\cup_{l=1}^{n_j} O_l^j$  whose union has measure  $|\cup_{l=1}^{n_j} O_l^j|$ .

4. We now fix an  $l \in \{1, \dots, n_j\}$ . Note that

$$U_l^j \subset O_l^j = \{z \in G_j'' \mid (Du(z), v_t(z)) \in \mathcal{G}_{l,\tau_j}^j\}$$

and that  $\mathcal{K}_l^j := \bar{\mathcal{G}}_{l,\tau_j}^j$  is a compact subset of  $\mathcal{S}_l^j = \mathcal{S}_{r_{k_l^j} - \bar{\mu}_{r_{k_l^j}}, r_{k_l^j} + \bar{\mu}_{r_{k_l^j}}}$ . Let  $\tilde{Q} \subset \mathbb{R}^n$  be a box with  $\Omega \subset \tilde{Q}$ , and let  $\tilde{I} = (0, T)$ . Applying Theorem 3.7 to box  $\tilde{Q} \times \tilde{I}$ ,  $\mathcal{K}_l^j \subset \subset \mathcal{S}_l^j$ , and  $\epsilon' = \frac{\epsilon |\Omega_T|}{20}$ , we obtain a constant  $\delta_l^j > 0$  that satisfies the conclusion of the theorem. By the uniform continuity of  $A$  on compact subsets of  $\mathbb{R}^n$ , we can find a  $\theta = \theta_{\epsilon, s_+} > 0$  such that

$$(5.4) \quad |A(p) - A(p')| < \frac{\epsilon}{20}$$

whenever  $|p|, |p'| \leq s_+$  and  $|p - p'| \leq \theta$ . Also by the uniform continuity of  $u, v$  and their gradients on  $\bar{U}_l^j$ , there exists a  $\nu_l^j > 0$  such that

$$(5.5) \quad \begin{aligned} & |u(z) - u(z')| + |\nabla u(z) - \nabla u(z')| + |v(z) - v(z')| \\ & + |\nabla v(z) - \nabla v(z')| < \min\{\frac{\delta_l^j}{2}, \frac{\epsilon}{20}, \theta\} \end{aligned}$$

whenever  $z, z' \in \bar{U}_l^j$  and  $|z - z'| \leq \nu_l^j$ . We now cover  $U_l^j$  (up to measure zero) by a sequence of disjoint boxes  $\{Q_{l,i}^j \times I_{l,i}^j\}_{i=1}^\infty$  in  $U_l^j$  with center  $z_{l,i}^j$  and diameter  $d_{l,i}^j < \nu_l^j$ .

5. Fix an  $i \in \mathbb{N}$ , and set  $w = (u, v)$  and  $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} = \nabla w(z_{l,i}^j) = \begin{pmatrix} Du(z_{l,i}^j) & u_t(z_{l,i}^j) \\ Dv(z_{l,i}^j) & v_t(z_{l,i}^j) \end{pmatrix}$ . By the choice of  $\delta_l^j > 0$  in Step 4 via Theorem 3.7, since  $Q_{l,i}^j \times I_{l,i}^j \subset \tilde{Q} \times \tilde{I}$  and  $(p, \beta) \in \mathcal{K}_l^j$ , for all sufficiently small  $\rho > 0$ , there exists a function  $\omega_{l,i}^j = (\varphi_{l,i}^j, \psi_{l,i}^j) \in C_c^\infty(Q_{l,i}^j \times I_{l,i}^j; \mathbb{R}^{1+n})$  satisfying

- (a)  $\operatorname{div} \psi_{l,i}^j = 0$  in  $Q_{l,i}^j \times I_{l,i}^j$ ,
- (b)  $(p' + D\varphi_{l,i}^j(z), \beta' + (\psi_{l,i}^j)_t(z)) \in \mathcal{S}_l^j$  for all  $z \in Q_{l,i}^j \times I_{l,i}^j$   
and all  $|(p', \beta') - (p, \beta)| \leq \delta_l^j$ ,
- (c)  $\|\omega_{l,i}^j\|_{L^\infty(Q_{l,i}^j \times I_{l,i}^j)} < \rho$ ,
- (d)  $\int_{Q_{l,i}^j \times I_{l,i}^j} |\beta + (\psi_{l,i}^j)_t(z) - A(p + D\varphi_{l,i}^j(z))| dz < \epsilon' |Q_{l,i}^j \times I_{l,i}^j| / |\tilde{Q} \times \tilde{I}|$ ,
- (e)  $\int_{Q_{l,i}^j \times I_{l,i}^j} \operatorname{dist}((p + D\varphi_{l,i}^j(z), \beta + (\psi_{l,i}^j)_t(z)), \mathcal{A}_l^j) dz < \epsilon' |Q_{l,i}^j \times I_{l,i}^j| / |\tilde{Q} \times \tilde{I}|$ ,
- (f)  $\int_{Q_{l,i}^j} \varphi_{l,i}^j(x, t) dx = 0$  for all  $t \in I_{l,i}^j$ ,
- (g)  $\|(\varphi_{l,i}^j)_t\|_{L^\infty(Q_{l,i}^j \times I_{l,i}^j)} < \rho$ ,

where the set  $\mathcal{A}_l^j = \mathcal{A}_{r_{k_l^j} - \bar{\mu}_{r_{k_l^j}}, r_{k_l^j} + \bar{\mu}_{r_{k_l^j}}} \subset \mathbb{R}^{n+n}$  is as in Theorem 3.7. Here,

we also let  $0 < \rho \leq \min\{\bar{\tau}_0, \frac{\delta_l^j}{2C}, \frac{\epsilon}{20C}, \eta\}$ , where  $C_n > 0$  is the constant in Theorem 2.4 and  $C$  is the product of  $C_n$  and the sum of the lengths of all sides of  $\tilde{Q}$ . From  $\varphi_{l,i}^j|_{\partial(Q_{l,i}^j \times I_{l,i}^j)} \equiv 0$  and (f), we can apply Theorem 2.4 to

$\varphi_{l,i}^j$  on  $Q_{l,i}^j \times I_{l,i}^j$  to obtain a function  $g_{l,i}^j = \mathcal{R}\varphi_{l,i}^j \in C^1(\overline{Q_{l,i}^j \times I_{l,i}^j}; \mathbb{R}^n) \cap W_0^{1,\infty}(Q_{l,i}^j \times I_{l,i}^j; \mathbb{R}^n)$  such that  $\operatorname{div} g_{l,i}^j = \varphi_{l,i}^j$  in  $Q_{l,i}^j \times I_{l,i}^j$  and

$$(5.6) \quad \|(g_{l,i}^j)_t\|_{L^\infty(Q_{l,i}^j \times I_{l,i}^j)} \leq C \|(\varphi_{l,i}^j)_t\|_{L^\infty(Q_{l,i}^j \times I_{l,i}^j)} \leq \frac{\delta_l^j}{2}. \quad (\text{by (g)})$$

6. As  $|v_t - A(Du)|$ ,  $\text{dist}((Du, v_t), \mathcal{B}) \in L^\infty(\Omega_T)$ , we can select a finite index set  $\mathcal{I} \subset \{(j, l) \mid j \in \mathbb{N}_0, 1 \leq l \leq n_j\} \times \mathbb{N} =: \mathcal{J}$  such that

$$(5.7) \quad \begin{cases} \int_{\bigcup_{(j,l,i) \in \mathcal{J} \setminus \mathcal{I}} Q_{l,i}^j \times I_{l,i}^j} |v_t(z) - A(Du(z))| dz \leq \frac{\epsilon}{5} |\Omega_T^1|, \\ \int_{\bigcup_{(j,l,i) \in \mathcal{J} \setminus \mathcal{I}} Q_{l,i}^j \times I_{l,i}^j} \text{dist}((Du(z), v_t(z)), \mathcal{B}) dz \leq \frac{\epsilon}{5} |\Omega_T^1|. \end{cases}$$

We finally define

$$(\tilde{u}, \tilde{v}) = (u, v) + \sum_{(j,l,i) \in \mathcal{I}} \chi_{Q_{l,i}^j \times I_{l,i}^j} (\varphi_{l,i}^j, \psi_{l,i}^j + g_{l,i}^j) \quad \text{in } \Omega_T.$$

7. Let us finally check that  $\tilde{u}$  together with  $\tilde{v}$  indeed gives the desired result. By construction, it is clear that  $\tilde{G} \subset \subset \Omega_T^1$ ,  $|\partial \tilde{G}| = 0$ ,  $\tilde{u} = u = u^*$  and  $\tilde{v} = v = v^*$  in  $\Omega_T \setminus \tilde{G}$ ,  $\tilde{u} \in C_{\text{piece}}^1(\tilde{G}) \cap W_{u^*}^{1,\infty}(\Omega_T)$ , and  $\tilde{v} \in C_{\text{piece}}^1(\tilde{G}; \mathbb{R}^n) \cap W_{v^*}^{1,\infty}(\Omega_T; \mathbb{R}^n)$ . By the choice of  $\rho$  in (g) as  $\rho \leq \bar{\tau}_0$ , we have  $\|\tilde{u}_t\|_{L^\infty(\Omega_T)} < m$ . Next, let  $(j, l, i) \in \mathcal{I}$ , and observe that for  $z \in Q_{l,i}^j \times I_{l,i}^j$ , with  $(p, \beta) = (Du(z_{l,i}^j), v_t(z_{l,i}^j)) \in \mathcal{G}_{l,\tau_j}^j$ , since  $|z - z_{l,i}^j| < d_{l,i}^j < \nu_i^j$ , it follows from (5.5) and (5.6) that

$$|(Du(z), v_t(z) + (g_{l,i}^j)_t(z)) - (p, \beta)| \leq \delta_l^j,$$

and so  $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}_l^j \subset \mathcal{S}_{dc}$  from (b) above. From (a) and  $\text{div } g_{l,i}^j = \varphi_{l,i}^j$ , for  $z \in Q_{l,i}^j \times I_{l,i}^j$ ,

$$\text{div } \tilde{v}(z) = \text{div}(v + \psi_{l,i}^j + g_{l,i}^j)(z) = u(z) + 0 + \varphi_{l,i}^j(z) = \tilde{u}(z).$$

Therefore,  $\tilde{u} \in \mathcal{U}$ . Next, observe

$$\begin{aligned} \int_{\Omega_T} |\tilde{v}_t - A(D\tilde{u})| dz &= \int_{\Omega_T \setminus \tilde{G}} |v_t^* - A(Du^*)| dz + \int_{\tilde{G}} |\tilde{v}_t - A(D\tilde{u})| dz \\ &= \int_{\Omega_T^0 \cup \Omega_T^2 \cup \Omega_T^3} |v_t^* - \tilde{A}(Du^*)| dz + \int_{\Omega_T^1 \setminus \tilde{G}} |v_t^* - A(Du^*)| dz + \int_{\tilde{G}} |\tilde{v}_t - A(D\tilde{u})| dz \\ &\leq \int_{\Omega_T^1 \setminus \tilde{G}} |v_t - A(Du)| dz + \sum_{j=0}^{\infty} \int_{G_j' \setminus G_j''} |v_t - A(Du)| dz \\ &\quad + \sum_{j=0}^{\infty} \sum_{l=1}^{n_j} \int_{F_l^j} |v_t - A(Du)| dz + \sum_{(j,l,i) \in \mathcal{J} \setminus \mathcal{I}} \int_{Q_{l,i}^j \times I_{l,i}^j} |v_t - A(Du)| dz \\ &\quad + \sum_{(j,l,i) \in \mathcal{I}} \int_{Q_{l,i}^j \times I_{l,i}^j} |\tilde{v}_t - A(D\tilde{u})| dz \\ &=: I_1^1 + I_2^1 + I_3^1 + I_4^1 + I_5^1, \\ \int_{\Omega_T} \text{dist}((D\tilde{u}, \tilde{v}_t), \mathcal{B}) dz &= \int_{\Omega_T \setminus \tilde{G}} \text{dist}((Du^*, v_t^*), \mathcal{B}) dz + \int_{\tilde{G}} \text{dist}((D\tilde{u}, \tilde{v}_t), \mathcal{B}) dz \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\Omega_T^1 \setminus \tilde{G}} \text{dist}((Du, v_t), \mathcal{B}) dz + \sum_{j=0}^{\infty} \int_{G_j' \setminus G_j''} \text{dist}((Du, v_t), \mathcal{B}) dz \\
&\quad + \sum_{j=0}^{\infty} \sum_{l=1}^{n_j} \int_{F_l^j} \text{dist}((Du, v_t), \mathcal{B}) dz + \sum_{(j,l,i) \in \mathcal{T} \setminus \mathcal{I}} \int_{Q_{l,i}^j \times I_{l,i}^j} \text{dist}((Du, v_t), \mathcal{B}) dz \\
&\quad + \sum_{(j,l,i) \in \mathcal{I}} \int_{Q_{l,i}^j \times I_{l,i}^j} \text{dist}((D\tilde{u}, \tilde{v}_t), \mathcal{B}) dz \\
&=: I_1^2 + I_2^2 + I_3^2 + I_4^2 + I_5^2.
\end{aligned}$$

From (5.1), (5.2), (5.3) and (5.7), we have  $I_1^k + I_2^k + I_3^k + I_4^k \leq \frac{4\epsilon}{5} |\Omega_T^1|$  ( $k = 1, 2$ ). Note that for  $(j, l, i) \in \mathcal{I}$  and  $z \in Q_{l,i}^j \times I_{l,i}^j$ , from (5.1), (5.2) and (g),

$$\begin{aligned}
|\tilde{v}_t(z) - A(D\tilde{u}(z))| &= |v_t(z) + (\psi_{l,i}^j)_t(z) + (g_{l,i}^j)_t(z) - A(Du(z) + D\varphi_{l,i}^j(z))| \\
&\leq |v_t(z) - v_t(z_{l,i}^j)| + |v_t(z_{l,i}^j) + (\psi_{l,i}^j)_t(z) - A(Du(z_{l,i}^j) + D\varphi_{l,i}^j(z))| \\
&\quad + |(g_{l,i}^j)_t(z)| + |A(Du(z_{l,i}^j) + D\varphi_{l,i}^j(z)) - A(Du(z) + D\varphi_{l,i}^j(z))| \\
&\leq \frac{\epsilon}{10} + |v_t(z_{l,i}^j) + (\psi_{l,i}^j)_t(z) - A(Du(z_{l,i}^j) + D\varphi_{l,i}^j(z))| \\
&\quad + |A(Du(z_{l,i}^j) + D\varphi_{l,i}^j(z)) - A(Du(z) + D\varphi_{l,i}^j(z))|.
\end{aligned}$$

Similarly, since  $\mathcal{A}_l^j \subset \mathcal{B}$ , we have

$$\begin{aligned}
&\text{dist}((D\tilde{u}(z), \tilde{v}_t(z)), \mathcal{B}) \\
&\leq \frac{\epsilon}{10} + \text{dist}((Du(z_{l,i}^j) + D\varphi_{l,i}^j(z), v_t(z_{l,i}^j) + (\psi_{l,i}^j)_t(z)), \mathcal{B}) \\
&\leq \frac{\epsilon}{10} + \text{dist}((Du(z_{l,i}^j) + D\varphi_{l,i}^j(z), v_t(z_{l,i}^j) + (\psi_{l,i}^j)_t(z)), \mathcal{A}_l^j).
\end{aligned}$$

From (b) and (i) of Theorem 3.5, we have  $|Du(z_{l,i}^j) + D\varphi_{l,i}^j(z)| \leq s_+(r_{k_l^j} + \bar{\mu}_{r_{k_l^j}}) < s_+$ . As  $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}_l^j$ , we also have  $|Du(z) + D\varphi_{l,i}^j(z)| = |D\tilde{u}(z)| < s_+$ , and by (5.5),  $|Du(z_{l,i}^j) - Du(z)| < \theta$ . From (5.4), we thus have

$$|A(Du(z_{l,i}^j) + D\varphi_{l,i}^j(z)) - A(Du(z) + D\varphi_{l,i}^j(z))| < \frac{\epsilon}{20}.$$

Integrating the two inequalities above over  $Q_{l,i}^j \times I_{l,i}^j$ , we now obtain from (d) and (e), respectively, that

$$\begin{aligned}
\int_{Q_{l,i}^j \times I_{l,i}^j} |\tilde{v}_t(z) - A(D\tilde{u}(z))| dz &\leq \frac{3\epsilon}{20} |Q_{l,i}^j \times I_{l,i}^j| + \frac{\epsilon |\Omega_T|}{20} \frac{|Q_{l,i}^j \times I_{l,i}^j|}{|\tilde{Q} \times \tilde{I}|} \leq \frac{\epsilon}{5} |Q_{l,i}^j \times I_{l,i}^j|, \\
\int_{Q_{l,i}^j \times I_{l,i}^j} \text{dist}((D\tilde{u}(z), \tilde{v}_t(z)), \mathcal{B}) dz &\leq \frac{\epsilon}{10} |Q_{l,i}^j \times I_{l,i}^j| + \frac{\epsilon |\Omega_T|}{20} \frac{|Q_{l,i}^j \times I_{l,i}^j|}{|\tilde{Q} \times \tilde{I}|} \leq \frac{\epsilon}{5} |Q_{l,i}^j \times I_{l,i}^j|;
\end{aligned}$$

thus  $I_5^k \leq \frac{\epsilon}{5} |\Omega_T^1|$ , and so  $I_1^k + I_2^k + I_3^k + I_4^k + I_5^k \leq \epsilon |\Omega_T^1|$ , where  $k = 1, 2$ . Therefore,  $\tilde{u} \in \mathcal{U}_\epsilon$ . Lastly, from (c) with  $\rho \leq \eta$  and the definition of  $\tilde{u}$ , we have  $\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta$ .

The proof is now complete.  $\square$

**5.2. Completion of the proof of Theorem 1.1.** Unless specifically distinguished, the proof below is common for both **Type I: FFT** solutions and **Type II: BFT** solutions.

*Proof of Theorem 1.1.* We return to Section 4. As outlined in Remark 4.4, Theorem 5.1 and Theorem 2.1 together give infinitely many Lipschitz solutions  $u$  to problem (1.1).

We now follow the proof of Theorem 2.1 for detailed information on such a Lipschitz solution  $u \in \mathcal{G}$  to (1.1). Here  $Du$  is the a.e.-pointwise limit of some sequence  $Du_j$ , where the sequence  $u_j \in \mathcal{U}_{1/j}$  converges to  $u$  in  $L^\infty(\Omega_T)$ . Since  $u_j \equiv u^*$  in  $\Omega_T \setminus G_j$  for some open set  $G_j \subset \subset \Omega_T^1$ , we also have  $u \equiv u^* \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_T^{\tilde{r}})$  so that

$$u_t = \operatorname{div}(A(Du)) \quad \text{and} \quad |Du| > s_+(\tilde{r}) \quad \text{in} \quad \Omega_T^{\tilde{r}} = \Omega_T^3$$

and

$$(5.8) \quad |Du| = 0 \quad \text{a.e. in} \quad \Omega_T^0, \quad \text{and} \quad |Du| = s_+(\tilde{r}) \quad \text{a.e. in} \quad \Omega_T^2.$$

Note  $(v_j)_t \rightharpoonup v_t$  in  $L^2(\Omega_T; \mathbb{R}^n)$ , where  $v_j$  is the corresponding vector function to  $u_j$  and  $v \in W^{1,2}((0, T); L^2(\Omega; \mathbb{R}^n))$ . From (2.2), we can even deduce that  $(v_j)_t \rightarrow v_t$  pointwise a.e. in  $\Omega_T$ . On the other hand, from the definition of  $\mathcal{U}_{1/j}$ ,

$$\int_{\Omega_T^1} \operatorname{dist}((Du_j, (v_j)_t), \mathcal{B}) \, dxdt \leq \frac{1}{j} |\Omega_T^1| \rightarrow 0 \quad \text{as } j \rightarrow \infty;$$

thus  $(Du, v_t) \in \mathcal{B}$  a.e. in  $\Omega_T^1$ , yielding together with (5.8) that

$$|Du| \in [s_-^2(\tilde{r}), s_+(\tilde{r})] \cup \{0\} \quad \text{in} \quad \Omega_T \setminus \Omega_T^{\tilde{r}}, \quad (\text{Type I})$$

$$|Du| \in [0, s_-^1(\tilde{r})] \cup [s_0, s_+(\tilde{r})] \quad \text{in} \quad \Omega_T \setminus \Omega_T^{\tilde{r}}, \quad (\text{Type II})$$

The proof is now complete.  $\square$

**5.3. Proof of Theorem 1.2.** Let  $u_0 \in C^{2+\alpha}(\bar{\Omega})$  with  $Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$ . If  $\|Du_0\|_{L^\infty(\Omega)} = 0$ , that is,  $u_0 \equiv c$  in  $\Omega$  for some constant  $c \in \mathbb{R}$ , the constant function  $u \equiv c$  in  $\Omega_T$  is a Lipschitz solution to problem (1.1). The existence of infinitely many Lipschitz solutions to (1.1) when  $|Du_0(x_0)| \in (0, s_+)$  for some  $x_0 \in \Omega$  is simply the result of Theorem 1.1. So we cover the remaining case here.

Assume  $\min_{\bar{\Omega}} |Du_0| \geq s_+$ . Fix any number  $0 < r = \tilde{r} < \sigma(s_+)$ , and let  $\tilde{\sigma}, \tilde{f} \in C^{1+\alpha}([0, \infty))$  be some functions from Lemma 2.3. Using the flux  $\tilde{A}(p) = \tilde{f}(|p|^2)p$ , Theorem 2.2 gives a unique solution  $u^* \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_T)$  to problem (2.4). If  $|Du^*|$  stays on or above the threshold  $s_+$  in  $\Omega_T$ , then  $u^*$  itself is a Lipschitz solution to (1.1). Otherwise, set  $\bar{s} = \frac{s_0 + s_+}{2}$  and choose a point  $(\bar{x}, \bar{t}) \in \Omega_T$  such that

$$|Du^*| \geq \bar{s} \quad \text{in} \quad \Omega \times (0, \bar{t}), \quad |Du^*(\bar{x}, \bar{t})| \in (0, s_+).$$



Regarding  $u_1(\cdot) := u^*(\cdot, \bar{t}) \in C^{2+\alpha}(\bar{\Omega})$ , satisfying  $Du_1 \cdot \mathbf{n}|_{\partial\Omega} = 0$ , as a new initial datum at time  $t = \bar{t}$ , it follows from Theorem 1.1 that problem (1.1), with the initial datum  $u_1$  at time  $t = \bar{t}$ , admits infinitely many Lipschitz solutions  $\bar{u}$  in  $\Omega \times (\bar{t}, T)$ . Then the patched functions  $u = \chi_{\Omega \times (0, \bar{t})} u^* + \chi_{\Omega \times [\bar{t}, T)} \bar{u}$  in  $\Omega_T$  become Lipschitz solutions to the original problem (1.1), and the proof is complete.

## 6. FURTHER REMARKS

In this final section, we briefly give an overview of how one can combine [9] with this paper to deduce further existence results. Instead of trying to formulate a certain general result, we present some case-by-case existence results in a casual manner. Also, even if one can possibly study the Dirichlet- and mixed-boundary value problems of forward-backward parabolic equations by using the methods of the two papers, we focus only on the Neumann problem (1.1) with mixed type (i.e., mixture of Perona-Malik, Höllig and non-Fourier types) profiles  $\sigma = \sigma(s) : [0, \infty) \rightarrow \mathbb{R}$  for diffusion fluxes  $A(p) = \frac{\sigma(|p|)}{|p|}p$ . Throughout this section, we assume that the domain  $\Omega$  and initial datum  $u_0$  satisfy condition (1.5). In addition, all the profiles  $\sigma(s)$  considered here are assumed to have derivative values lying in some interval  $[\lambda, \Lambda]$  ( $\Lambda \geq \lambda > 0$ ) for all sufficiently large  $s > 0$ .

As an example, consider the profile  $\sigma(s)$  given by the following graph.

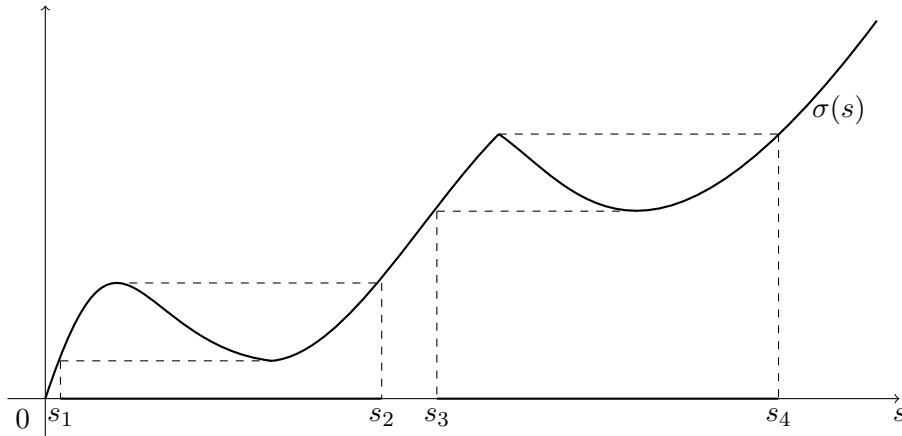


FIGURE 5. The first example of profile  $\sigma(s)$ .

For such a profile  $\sigma(s)$ , if the initial gradient size  $|Du_0|$  belongs to the *phase transition zones*  $(s_1, s_2)$  and  $(s_3, s_4)$  at some points in  $\Omega$ , one can employ the method of [9] to generate solutions. In doing so, one may choose a modified profile  $\tilde{\sigma}(s)$  that is equal to the original  $\sigma(s)$  outside  $(s_1, s_2) \cup (s_3, s_4)$  and whose derivative values always belong to some interval  $[\theta, \Theta]$  ( $\Theta \geq \theta > 0$ ). Then obtained solutions will be smooth evolutions in the subdomain of

$\Omega_T$  in which a certain classical solution  $u^*$  corresponding to the profile  $\tilde{\sigma}(s)$  with initial datum  $u_0$  has gradient size lying outside  $(s_1, s_2) \cup (s_3, s_4)$ . In the subdomain of  $\Omega_T$  in which  $|Du^*|$  lies in two fixed disjoint open intervals in  $(s_1, s_2) \cup (s_3, s_4)$ , where we do *laminare* (i.e., convexify in the rank-one sense) certain matrix sets to capture some open structures that enable us to do a *surger*, such solutions will be highly oscillatory as those should have gradient size that belongs only to four disjoint intervals. The only difference from [9] in getting solutions is that we formulate two matrix sets to be laminated instead of one. Obviously, finitely many ups and downs of the graph of a profile  $\sigma(s)$  can be also dealt in the same way for existence as long as the graph is increasing like a staircase as in Figure 5. However, profiles of non-staircase shape are still possible to handle as we explain below.

We next consider the profile  $\sigma(s) = \sin s$  whose graph is given below. Assume in this case that the space domain  $\Omega$  is convex to guarantee the gradient maximum principle for uniformly parabolic diffusions [8, Theorem 2.1].

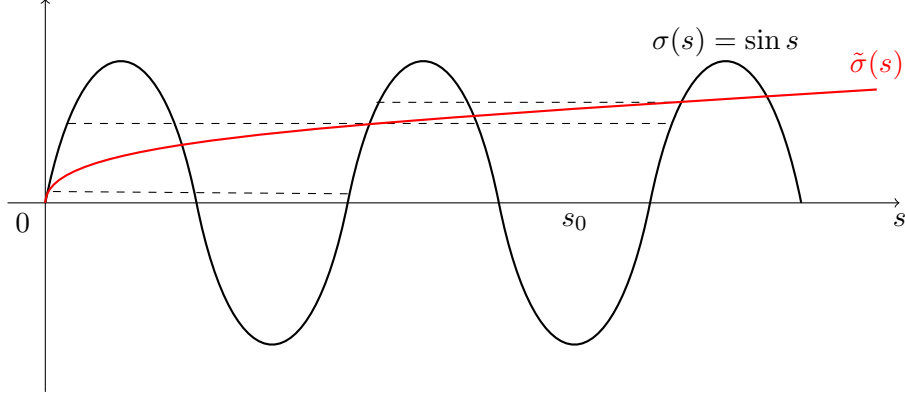


FIGURE 6. The second example of profile  $\sigma(s)$ .

For instance, suppose  $3\pi < s_0 := \sup_{\Omega} |Du_0| < 4\pi$ ; then select a suitable modified profile  $\tilde{\sigma}(s)$  as in Figure 6. Following [9], we can perform (simultaneous) lamination for two matrix sets to obtain solutions whose gradient size belongs to several disjoint intervals. Here, the gradient maximum principle for the classical solution  $u^*$  corresponding to  $\tilde{\sigma}(s)$  with initial datum  $u_0$  is required to construct an admissible class containing a pair of functions with  $u^*$  as its first component. If  $|Du^*|$  were escaping beyond  $s_0$ , we would not be able to define the admissible class to problem (1.1) for any given  $T > 0$ . In case of profile  $\sigma(s) = -\sin s$ , non-Fourier type diffusion may arise in the small gradient regime where  $|Du^*| < 3\pi/2$ ; thus the method of this paper can be combined with that of [9] to obtain solutions for such a profile  $\sigma(s)$ .

Lastly, consider the profile  $\sigma(s)$  having the graph as below. In this case, non-Fourier type diffusion may occur in the interval  $(0, s_+)$ , and Höllig type

phase transitions arise in the interval  $(s_1, s_2)$ . Clearly, the former type can be handled by the technique of this paper, and the latter by that of [9]. One essential difference between these two techniques is that although the one in the current paper stems from the other in [9], it makes use of countably many partial rank-one structures of the related matrix set to avoid a certain *degeneracy* which occurs at the points  $x \in \Omega$  for which  $\sigma(|Du_0(x)|) = 0$  when  $t = 0$ . (Such a degeneracy does not appear in space dimension  $n = 1$ .) Accordingly, compared to a single partial rank-one structure used in [9], more sophisticated scheme has been used to detect partial rank-one structures and to combine those in this paper.

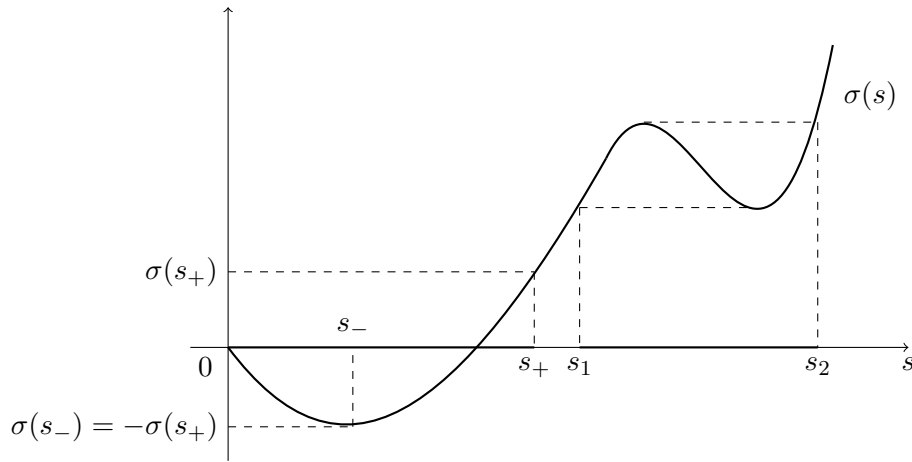


FIGURE 7. The third example of profile  $\sigma(s)$ .

In concluding this paper, we remark that for many other possible profiles  $\sigma(s)$  that are a combination of Perona-Malik, Höllig and non-Fourier types, one can expect the existence of solutions in *almost* all cases.

#### REFERENCES

- [1] N. Alikakos and R. Rostamian, *Gradient estimates for degenerate diffusion equations. I*, Math. Ann., **259** (1) (1982), 53–70.
- [2] J. Bourgain and H. Brezis, *On the equation  $\operatorname{div} Y = f$  and application to control of phases*, J. Amer. Math. Soc., **16** (2) (2002), 393–426.
- [3] H. Brézis, “Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert,” North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.
- [4] A. Bruckner, J. Bruckner and B. Thomson, “Real analysis,” Prentice-Hall, 1996.
- [5] B. Dacorogna, “Direct methods in the calculus of variations,” Second edition. Applied Mathematical Sciences, 78. Springer, New York, 2008.
- [6] W. Day, “The thermodynamics of simple materials with fading memory,” Tracts in Natural Philosophy, 22. Springer-Verlag, New York, Heidelberg and Berlin, 1970.

- [7] K. Höllig, *Existence of infinitely many solutions for a forward backward heat equation*, Trans. Amer. Math. Soc., **278** (1) (1983), 299–316.
- [8] S. Kim and B. Yan, *Convex integration and infinitely many weak solutions to the Perona-Malik equation in all dimensions*, SIAM J. Math. Anal., **47** (4) (2015), 2770–2794.
- [9] S. Kim and B. Yan, *On Lipschitz solutions for some forward-backward parabolic equations*, Preprint.
- [10] O.A. Ladyženskaja and V.A. Solonnikov and N.N. Ural’ceva, “Linear and quasilinear equations of parabolic type. (Russian),” Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. 1968.
- [11] G.M. Lieberman, “Second order parabolic differential equations,” World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
- [12] P. Perona and J. Malik, *Scale space and edge detection using anisotropic diffusion*, IEEE Trans. Pattern Anal. Mach. Intell., **12** (1990), 629–639.
- [13] C. Truesdell, “Rational thermodynamics,” 2nd ed., Springer-Verlag, New York, 1984.
- [14] K. Zhang, *Existence of infinitely many solutions for the one-dimensional Perona-Malik model*, Calc. Var. Partial Differential Equations, **26** (2) (2006), 171–199.

INSTITUTE FOR MATHEMATICAL SCIENCES, RENMIN UNIVERSITY OF CHINA, BEIJING 100872, PRC

*E-mail address:* kimseo14@ruc.edu.cn

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824, USA

*E-mail address:* yan@math.msu.edu